Global asymptotic stability in Volterra's population systems

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Abstract. Sufficient conditions which can be verified easily are obtained for the global asymptotic stability of the positive steady state in Volterra's population system incorporating hereditary effects.

Key words: global asymptotic stability — hereditary effects — persistence — coexistence — positive limit set

1. Introduction

One of the simplest models of the dynamics of $n$-interacting species has been proposed by Volterra (1931) in the form of an autonomous system of $n$ ordinary differential equations with quadratic nonlinearities,

$$\frac{dx_i(t)}{dt} = x_i(t) \left\{ b_i + \sum_{j=1}^{n} a_{ij}x_j(t) \right\} \quad i = 1, 2, \ldots, n \quad (1.1)$$

where $b_i, a_{ij} \ (i, j = 1, 2, \ldots, n)$ are constants and $x_i(t)$ denotes the population size (or biomass) of the $i$th species at time $t$. Although it has been said that the Volterra model (1.1) is too simple to reflect ecological processes realistically, it is still worthwhile to study the global properties of (1.1) especially for $n \geq 3$ since (1.1) is a simple nonlinear system exhibiting a rich collection of interesting phenomena.

One of the attempts aimed at a realistic improvement of the simple model (1.1) is to incorporate discrete or continuous time delays (hereditary or historical actions) in (1.1). Volterra (1931) himself has proposed such a model in the form of an integrodifferential system

$$\frac{dx_i(t)}{dt} = x_i(t) \left\{ b_i + \sum_{j=1}^{n} a_{ij}x_j(t) \right\}$$

$$+ \sum_{j=1}^{n} b_{ij} \int_{-\infty}^{t} F_{ij}(t-s)x_j(s) \, ds \right\} \quad i = 1, 2, \ldots, n \quad (1.2)$$
where \( b_i, a_{ij}, b_j \) (\( i,j = 1,2, \ldots, n \)) are real constants and \( F_{ij} :[0, \infty) \to [0, \infty) \) are continuous functions normalised such that

\[
\int_{0}^{\infty} F_{ij}(s) \, ds = 1; \quad i,j = 1,2, \ldots, n. \tag{1.3}
\]

A special case of (1.2) for \( n = 2 \) has been discussed by Volterra (1931) (see for instance Scudo and Ziegler (1978)). Recently Wörz-Busekros (1978) has obtained a set of sufficient conditions for the global asymptotic stability of systems of the form (1.2)–(1.3) assuming that the delay kernels \( F_{ij} \) are convex combinations of the functions

\[
F_m(t) = \frac{a^m}{(m-1)!} t^{m-1} e^{-at}; \quad m = 1,2, \ldots; a > 0 \text{ is a constant} \tag{1.4}
\]

\( F_0(t) = 0. \)

Systems like (1.2)–(1.4) can be reduced to a higher order system of autonomous ordinary differential equations due to the special nature of \( F_{ij} \) in (1.4) (see MacDonald, 1978). In an elaborate discussion, Cushing (1977) has considered various aspects of local stability and bifurcation to oscillations in integro-differential equations.

Recently Gopalsamy (1980) has discussed the asymptotic behaviour of the dynamics of two species competition described by a Volterra system of the form

\[
\frac{dx}{dt} = x \left\{ r_1 - \alpha_{11} x - \alpha_{12} \int_{0}^{T} k_{12}(s) y(t-s) \, ds \right\}
\]

\[
\frac{dy}{dt} = y \left\{ r_2 - \alpha_{21} \int_{0}^{T} k_{21}(s) x(t-s) \, ds - c_{22} y \right\} \tag{1.5}
\]

where the delay kernels \( k_{12}, k_{21} \) are continuous, positive and satisfy

\[
\int_{0}^{T} k_{12}(s) \, ds = 1 = \int_{0}^{T} k_{21}(s) \, ds. \tag{1.6}
\]

Assuming that the constants \( \alpha_{ij} \) (\( i,j = 1,2 \)) in (1.5) are positive such that

\[
\frac{\alpha_{11}}{\alpha_{21}} > \frac{r_1}{r_2} > \frac{\alpha_{12}}{\alpha_{22}} \tag{1.7}
\]

and using an iterative technique, it has been shown by Gopalsamy (1980) that all solutions of (1.5)–(1.7) with positive initial conditions on \([-T,0]\) have the property \( \{x(t), y(t)\} \to (x^*, y^*) \) as \( t \to \infty \) where \( (x^*, y^*) \) is the unique positive steady state of (1.5)–(1.7) given by \( \alpha_{11} x^* + \alpha_{12} y^* = r_1; \alpha_{21} x^* + \alpha_{22} y^* = r_2. \)

In the following we consider a more general class of \( n \)-species systems and obtain sufficient conditions for the global asymptotic stability of such systems. Unlike the previous work (Gopalsamy, 1980) the analysis is not dependent on the precise nature of the ecological association of the \( n \)-species; we also examine a closely related problem of "persistence" of an \( n \)-species system and derive a necessary condition for the "persistence" of a system described by integro-differential equations.