The Theory of Matrix Polynomials and its Application to the Mechanics of Isotropic Continua

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Abstract

In this paper we show that a symmetric isotropic matrix polynomial in any number of symmetric $3 \times 3$ matrices can be expressed as a symmetric isotropic matrix polynomial, in which each of the matrix products is formed from at most six matrices and has one of a certain number of forms which are explicitly given. The significance of these results in the mechanics of isotropic continua is indicated.

1. Introduction

We consider a body to undergo deformation described in a rectangular Cartesian coordinate system $x$ by

$$x_i = x_i(X_j, t),$$

where $x_i$ and $X_i$ are the coordinates in the system $x$ of a generic particle of the body at arbitrary time $t$ and a standard reference time respectively. We assume that the stress components $t_{ij}$ at the point $x_i$ and time $t$ are single-valued functions of the deformation gradients $\partial x_p/\partial X_q$ and of the gradients of the velocity $v^{(1)}$, the acceleration $v^{(2)}$, ..., the $(n-1)$th acceleration $v^{(n)}$, at that point and time; i.e. the constitutive equation takes the form

$$t_{ij} = t_{ij}(\frac{\partial x_p}{\partial X_q}, \frac{\partial v^{(1)}_p}{\partial x_q}, ..., \frac{\partial v^{(n-1)}_p}{\partial x_q}).$$  (1.1)
It has been shown [1] that if, further, the material is isotropic at the standard reference time, the stress components \( t_{ij} \) must be expressible as single-valued functions of the quantities \( C_{pq} \) and \( A_{pq}^{(r)} (r = 1, 2, \ldots, n) \) defined by

\[
C_{pq} = \frac{\partial x_p}{\partial \xi_m} \frac{\partial x_q}{\partial \xi_m}, \quad A_{pq}^{(1)} = \frac{\partial v_{ij}^{(1)}}{\partial x_q} + \frac{\partial v_{ij}^{(1)}}{\partial x_p},
\]

and

\[
A_{pq}^{(r)} = \frac{DA_{pq}^{(r)}}{Dt} + A_{pq}^{(r-1)} \frac{\partial v_{ij}^{(1)}}{\partial x_q} + A_{pq}^{(r-1)} \frac{\partial v_{ij}^{(1)}}{\partial x_p},
\]

where \( D/Dt \) denotes the material time derivative; i.e. the constitutive equation takes the form

\[
t_{ij} = t_{ij}(C_{pq}, A_{pq}^{(1)}, A_{pq}^{(2)}, \ldots, A_{pq}^{(n)}),
\]

where the functional dependence of \( t_{ij} \) on the arguments may be different from that in equation (1.1).

Now, \( C_{pq} \) and \( A_{pq}^{(r)} \) have the property that, if quantities \( \bar{C}_{pq} \) and \( \bar{A}_{pq}^{(r)} \) are defined in an analogous manner in any other rectangular Cartesian system \( \bar{x} \) moving in an arbitrary manner with respect to \( x \), then \( \bar{C}_{pq} \) and \( C_{pq} \) and \( \bar{A}_{pq}^{(r)} \) and \( A_{pq}^{(r)} \) are the components in the systems \( \bar{x} \) and \( x \) respectively of Cartesian tensors.

We now define the symmetric \( 3 \times 3 \) matrices \( T, C \) and \( A_r \) by

\[
T = ||t_{ij}||, \quad C = ||C_{ij}|| \quad \text{and} \quad A_r = ||A_{ij}^{(r)}||.
\]

It has been shown [1] that, if we assume the dependence of \( t_{ij} \) on the arguments in (1.3) to be polynomial, it follows from the isotropy of the material that the stress matrix \( T \) must be expressible as a symmetric isotropic matrix polynomial in the kinematic matrices \( C \) and \( A_r (r = 1, 2, \ldots, n) \), the coefficients in which are polynomial scalar invariants under orthogonal transformations of the matrices \( C \) and \( A_r (r = 1, 2, \ldots, n) \).

If only two kinematic matrices occur in the isotropic matrix polynomial expression for \( T \), as would be the case if in (1.1) \( t_{ij} \) were taken to be a function of \( \partial x_p/\partial x_q \) and \( \partial v_{ij}^{(1)}/\partial x_q \) only, or of \( \partial v_{ij}^{(1)}/\partial x_p \) and \( \partial v_{ij}^{(1)}/\partial x_q \) only, then \( T \) may be expressed in a closed form [2]. For example, if the argument matrices are \( A_1 \) and \( A_2 \), \( T \) may be expressed in the form

\[
T = \psi_0 I + \psi_1 A_1 + \psi_2 A_2 + \psi_3 A_1^2 + \psi_4 A_2^2 + \psi_5 (A_1 A_2 + A_2 A_1) +
+ \psi_6 (A_1^2 A_2 + A_2^2 A_1) + \psi_7 (A_1 A_2^2 + A_2 A_1^2) + \psi_8 (A_1^2 A_2^2 + A_2^2 A_1^2),
\]

where the \( \psi \)'s are polynomial invariants, under orthogonal transformations of the matrices \( A_1 \) and \( A_2 \) and may therefore be expressed as polynomials in the integrity basis \( tr A_1, tr A_2, tr A_1^2, tr A_2^2, tr A_1 A_2, tr A_1^2 A_2, tr A_2^2 A_1, tr A_1 A_2^2, tr A_2 A_1^2 \).

In the present paper, we consider the more general case when the stress matrix \( T \) is an isotropic matrix polynomial in an arbitrary number of kinematic matrices \( C, A_1, \ldots, A_n \) and derive an expression for \( T \) as an isotropic matrix polynomial of closed form. The final result is given by Theorem 5 in § 10, by substituting \( C, A_1, A_2, \ldots, A_n \) for \( a_p (P = 1, 2, \ldots, R) \).

In arriving at Theorem 5, the following procedure is adopted. In § 2, following a statement of certain definitions, we derive, essentially from the Hamilton—