Coldness and Temperature

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1. Introduction

A rational thermodynamic theory has been developed by I. Müller during the last four years [1, 2, 3]. It is not of the after effect type. Rather, it is restricted to constitutive equations which do not involve the histories of the independent variables but may involve these variables and their time derivatives up to some order, all taken at the present time. A derived quantity in this theory is the coldness function, which is universal for a certain class of materials. It is certainly worthwhile to study the physical significance of this function. We shall do this for the simplest case, which is that of pure heat conduction in an isotropic rigid material.

We shall start from a generalization of Müller’s theory which has recently been obtained by Batra [4]. In this theory it is assumed that the constitutive assumptions express the various dependent quantities at \( x, t \) in terms of \( \mathcal{H}(x, t) \) and \( \mathcal{A}(x, t) \) where \( \mathcal{H}(x, t) \) is designated as the temperature field. First we examine the situation from a purely mathematical point of view, and then in the light of the results we shall discuss to what extent the interpretation of \( \mathcal{H}(x, t) \) as a temperature field seems reasonable.

2. A Mathematical Problem

First we present Müller’s idea and results in Batra’s generalization. However, we shall do so in a formal way, and the notation should not conceal the fact that this is pure mathematics.

A density \( \varepsilon(x, t) \) and an associated flux \( q(x, t) \) are introduced and are postulated to satisfy the equation

\[
\dot{\varepsilon} + \nabla \cdot q = 0.
\]  

(2.1)

Also a density \( \eta(x, t) \) and an associated flux \( \Phi(x, t) \) are introduced and are postulated to satisfy the inequality

\[
\dot{\eta} + \nabla \cdot \Phi \geq 0.
\]

(2.2)

These four fields are linked together by the constitutive assumption that they depend on position and time through a field \( \mathcal{H}(x, t) \), in the following way.
First \( \eta \)-postulate: The fields \( \varepsilon, \eta, q, \Phi \) are continuous isotropic functions of the variables \( \vartheta, \dot{\vartheta}, \ldots, \vartheta^{(N)}, V \vartheta \) with partial derivatives up to second order

\begin{align}
\varepsilon &= \varepsilon(\vartheta, \dot{\vartheta}, \ldots, \vartheta^{(N)}; g), \quad q = -\kappa(\vartheta, \dot{\vartheta}, \ldots, \vartheta^{(N)}; g) V \vartheta, \\
\eta &= \eta(\vartheta, \dot{\vartheta}, \ldots, \vartheta^{(N)}; g), \quad \Phi = \varphi(\vartheta, \dot{\vartheta}, \ldots, \vartheta^{(N)}; g) V \vartheta,
\end{align}

where

\[ g = V \vartheta \cdot V \vartheta. \]

Whenever \((2.1)\) holds with unrestricted values of

\[ \dot{\vartheta}, \ldots, \vartheta^{(N)}, \vartheta^{(N+1)}; \quad V \vartheta, V \dot{\vartheta}, \ldots, V \vartheta^{(N)}; \quad \vartheta^{2} \vartheta/\vartheta_{A} \vartheta_{B}, \]

then the inequality \((2.2)\) holds with the equality sign being assumed in equilibrium only, that is for \( \vartheta^{(n)} = 0 \) \((n = 1, 2, \ldots, N)\), \( g = 0 \).

Derivatives of functions with respect to \( \vartheta^{(n)} \) will be indicated by a subscript \( n \) and the subscripts \( 0, g \) denote the derivatives \( \vartheta/\vartheta_{0}, \vartheta/\vartheta_{g} \), respectively. The symbol \( |_{E} \) indicates that the expression to its left must be taken at equilibrium.

Now a few assumptions are introduced.

A1:

\[ \kappa > 0. \quad (2.6) \]

We are not interested in the case in which \( \kappa = 0 \), so we assume \( \kappa \neq 0 \) for all values of \( \vartheta^{(N)} \) \((n = 1, 2, \ldots, N)\), of \( g \), and for all values of \( \vartheta \) in a certain finite range. If \( \kappa \) were negative, we could replace the field \( \vartheta(x, t) \) by the field \( \tilde{\vartheta}(x, t) = -\vartheta(x, t) \) and \( \kappa \) by \( -\kappa \) and formally the same equations \((2.3), (2.4), (2.5)\) would result with a positive value of \( \tilde{\kappa} = -\kappa \):

A2:

\[ \varepsilon_{0}|_{E} \neq 0, \quad \eta_{0}|_{E} \neq 0 \quad (2.7) \]

in the range of \( \vartheta \). These assumptions guarantee that \( \varepsilon \) and \( \eta \) depend genuinely on \( \vartheta \) and that there are no stationary points. The condition \((2.1)\) can be ignored if one introduces a kind of Lagrange parameter \( \Lambda \) and requires that

\[ \dot{\vartheta} - \Lambda \dot{\vartheta} + V \cdot \Phi - \Lambda V \cdot q \geq 0 \quad (2.8) \]

for all \( \vartheta(x, t) \) fields and not only for those restricted by \((2.1)\) (see MÜLLER \([2, 3]\)).

It can also be shown that \( \Lambda \) is a function of \( \vartheta, \dot{\vartheta}, \ldots, \vartheta^{(N)}, g \). Then, by the algebra which is given in detail by MÜLLER and BATRA, the following necessary and sufficient conditions are obtained:

\[ \eta_{N} - \Lambda \varepsilon_{N} = 0, \quad \eta_{0} - \Lambda \varepsilon_{0} = \frac{1}{2} \kappa \Lambda_{1}, \quad \varphi + \Lambda \kappa = 0, \quad \Lambda = \Lambda(\vartheta, \dot{\vartheta}), \quad (2.9) \]

\[ \sum_{n=0}^{N-1} (\eta_{n} - \Lambda \varepsilon_{n}) \vartheta^{(n+1)} - \Lambda_{0} \kappa g \geq 0. \quad (2.10) \]

Cross differentiation of \((2.9)_{1, 2}\) yields

\[ -2\Lambda_{N} \varepsilon_{z} = \Lambda_{1} \kappa_{N} + \Lambda_{1N} \kappa. \quad (2.11) \]

The inequality \((2.10)\) must hold for all values of \( \vartheta^{(n)} \) \((n = 0, 1, \ldots, N)\) and all \( g \geq 0 \). The minimum of the left member is zero which is attained for \( \vartheta^{(n)} = 0 \).