Continuum Limits of Discrete Gases

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1. Introduction

In 1971 Greenberg & Hedstrom [1] investigated the limiting behavior of a discrete interacting particle system as the initial particle spacing and masses tend to zero. The model studied assumed a countable collection of particles, each with mass $\tilde{\rho}h$, moving along the real axis according to

$$\tilde{\rho}h\dot{x}_k = 0, \quad \dot{x}_k(0) = u_0(kh), \quad \text{and} \quad x_k(0) = kh,$$

(1.1)

for $k = 0, \pm 1, \pm 2, \ldots$ When particles collide, momenta and energy are conserved. This condition leads to the following interaction rule: if particles $p, p + 1, \ldots, p + q$ collide at some instant and if $v_p^{-} > v_{p+1}^{-} > \ldots > v_{p+q}^{-}$ are the particle velocities before the collision, then the velocities after the collision, which we denote by $(v_p^{+}, v_{p+1}^{+}, \ldots, v_{p+q}^{+})$, are given by $(v_p^{+}, v_{p+1}^{+}, \ldots, v_{p+q}^{+}) = (v_p^{-}, v_{p+1}^{-}, \ldots, v_{p+q}^{-})$.

The principal motivation behind the Greenberg & Hedstrom’s investigation was to establish the existence of the limiting continuum motion

$$z(x, t) \overset{\text{def}}{=} \lim_{h \to 0, \ \text{kh=x \ fixed}} x_k(t)$$

(1.2)

and to examine regularity properties of this limit. No limiting equation of motion was obtained for $z$ though it was shown that the limit motion did exhibit shock waves as boundaries of regions where the particle motions were highly oscillatory.

In this note we shall again examine the problem discussed in [1]. What is new is the simpler characterization of the limiting field quantities such as density ($\hat{\rho}$), velocity ($\hat{u}$), and pressure ($\hat{p}$), as well as a characterization of the equations satisfied by these fields. In certain cases we obtain a closed system of conservation laws for $\hat{\xi}$, $\hat{u}$, and $\hat{p}$.

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1 In that note we restricted our attention to velocity data $\xi \to u_0(\xi) = \begin{cases} u_l, & \xi < 0 \\ u_r, & \xi > 0 \end{cases}$ where $u_l > u_r$. Other choices would have lead to similar results.
2. The Model

We let \( f \) be the one-particle distribution function and observe that for particles of equal mass travelling on the real axis \( f \) satisfies the free stream equation

\[
\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} = 0. \tag{2.1}
\]

The absence of collision terms on the right hand side of (2.1) obtains from the fact that if two particles with velocities \( v_1, v_2 \) collide, then these particles emerge with velocities \( (v_1^+, v_2^+) = (v_2^-, v_1^-) \) and thus the number of particles with a given velocity is conserved in a collision. Our interest is in solutions of (2.1) satisfying the initial condition that \( f \) is Maxwellian; i.e.

\[
f(x, v, 0) \overset{\text{def}}{=} \frac{q_0(x)}{\sqrt{2\pi\varepsilon}} \exp \left( -\frac{(v - u_0(x))^2}{2\varepsilon} \right). \tag{2.2}
\]

The functions \( x \mapsto q_0(x) \gtrless q \) and \( x \mapsto u_0(x) \) represent the particle density and mean velocity at \( x \) at \( t = 0 \) and the number \( 0 < \varepsilon \ll 1 \) is the assumed uniform temperature at \( t = 0 \); that is

\[
(q_0, q_0u_0)(x) = \int_{-\infty}^{\infty} (1, v) \frac{q_0(x) \exp \left( -\frac{(v - u_0(x))^2}{2\varepsilon} \right)}{\sqrt{2\pi\varepsilon}} \, dv. \tag{2.3}
\]

The initial value of the pressure, \( p_0(x, \varepsilon) \), is given by

\[
p_0(x, \varepsilon) = \int_{-\infty}^{\infty} (v - u_0(x))^2 \frac{q_0(x) \exp \left( -\frac{(v - u_0(x))^2}{2\varepsilon} \right)}{\sqrt{2\pi\varepsilon}} \, dv = q_0(x) \varepsilon. \tag{2.4}
\]

The one-particle distribution function at future times is given by

\[
f(x, t, v) = \frac{q_0(x - vt) \exp \left( -\frac{(v - u_0(x - vt))^2}{2\varepsilon} \right)}{\sqrt{2\pi\varepsilon}}, \tag{2.5}
\]

and the moments of interest to us are given by

\[
(q, qu, p, q)(x, t, \varepsilon)
= \int_{-\infty}^{\infty} (1, v, (v - u(x, t, \varepsilon))^2, (v - u(x, t, \varepsilon))^3) \frac{q_0(x - vt) \exp \left( -\frac{(v - u_0(x - vt))^2}{2\varepsilon} \right)}{\sqrt{2\pi\varepsilon}} \, dv
= \int_{-\infty}^{\infty} \left(1, \frac{x - \xi}{t}, \left(\frac{x - \xi}{t} - u(x, t, \varepsilon)\right)^2, \left(\frac{x - \xi}{t} - u(x, t, \varepsilon)\right)^3\right) \frac{q_0(\xi) \exp \left( -\frac{(x - \xi - u_0(\xi) t)^2}{2\varepsilon t^2} \right)}{\sqrt{2\pi\varepsilon t}} \, d\xi. \tag{2.6}
\]