Finite Amplitude One-dimensional Pulses in an Inhomogeneous Granular Material

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1. Introduction

Following the work of Cowin & Goodman [1, 2, 3] in deriving a continuum theory for description of the response of many granular materials, like sands, grains and pressed powders, Nunziato & Walsh have investigated some aspects of wavelike behavior in such materials. In [4] they linearize the equations and discuss dispersive effects, which are important in disturbances having a ‘wavelength’ comparable with the natural length defined by the medium. In [5, 6], using the equations developed by Cowin & Goodman, they discuss one-dimensional acceleration waves. Although this is an exact theory, its predictions concern jumps in acceleration and in strain gradient defined only at a wavefront. These are immeasurable quantities, since the governing equations are hyperbolic, and information carried along characteristics arbitrarily close to the wavefront propagates independently. Consequently, acceleration wave theory should be regarded as relevant only to the mathematical limit of infinitesimally short disturbances.

In this paper we show how modulated simple wave theory [7-10] is no harder to derive and yields exact equations for signal growth which are generalizations of those in acceleration wave theory. We outline an iterative procedure based on the premise that a ‘wavelength’ of a pulse is short compared to length scales defined by inhomogeneities and by the natural length of the medium. At first iteration it yields two ordinary differential equations along each characteristic. One determines the modulation of strain (hence density, velocity, etc.), while the second governs wavelet spacing. This second equation describes the effects of amplitude dispersion, pulse distortion and shock formation.

In section 5 we discuss the small amplitude, finite rate theory [9, 10, 11]. In this case both ordinary differential equations are linear. Together they are equivalent to the Bernoulli equation which governs the modulation of jump discontinuities in acceleration wave theory. In particular, we derive a simple formula for the modulation of strain, showing that along each wavelet the strain is a function only of local material properties. Hence the two material coef-
ficients $\mu_0$, $\kappa_0$ discussed by NUNZIATO & WALSH have simple physical interpretations and may be related to the modulation of an entire pulse, not just to wavefront discontinuities.

For rapid pulses of finite amplitude the two modulation equations are no longer linear. However they may be integrated, as in section 6, to give a representation of the modulation of a rapid pulse which generalizes that of small amplitude, finite rate theory. As in that theory, predictions of the occurrence and location of shock formation is given. Section 7 treats the propagation of a pulse into a uniform region. Small amplitude, finite rate theory predicts that the pulse propagates as a simple wave, with no detectable void compaction. Rapid pulses of finite amplitude also propagate essentially as simple waves, but void compaction is then seen to be of third order in time elapsed since the passage of the wavefront.

2. Background Theory

In the continuum theory of granular media developed in [1–3], one-dimensional disturbances are described by the current position $x = x(X, t)$ of material having reference position $X$, and by a quantity $v = v(X, t)$ known as the \textit{volume distribution function}. The granules have density $\gamma = \gamma(X, t)$, and the bulk density is $\rho = \gamma v = \rho(X, t)$. The reference values of $v$, $\gamma$ and $\rho$ are $v_0 = v_0(X)$, $\gamma_0 = \gamma_0(X)$, $\rho_0 = \gamma_0 v_0 = \rho_0(X)$, respectively. In the absence of heat conduction the governing equations are (see [5])

\begin{align}
\rho_0 x_{tt} &= T_x + \rho_0 b, \quad (2.1) \\
\rho_0 k v_{tt} &= h_x + \rho_0 g, \quad (2.2)
\end{align}

where $T$ is the stress, $b$ is the usual extrinsic body force, and $k$, $h$ and $g$ are known respectively as the \textit{equilibrated inertia}, the \textit{equilibrated stress} and the \textit{intrinsic equilibrated body force}. In [5] NUNZIATO & WALSH allow $k$ to depend on $X$ and assume that the quantities $T$, $h$ and $g$ are derivable from the specific internal energy

\begin{align}
e &= \hat{e}(v_0, v, v_X, f; \eta), \quad f \equiv \gamma_0/\gamma, \quad (2.3)
\end{align}

where $\eta = \eta(X)$ is the specific entropy, as

\begin{align}
T &= \rho f \hat{e}_f, \quad (2.4) \\
g &= -\frac{T}{\rho v} \hat{e}_v, \quad (2.5)
\end{align}

and

\begin{align}h &= \rho_0 \hat{e}_{v_X}. \quad (2.6)
\end{align}

We find it preferable to work in terms of $F$ and $E$, where

\begin{align}F &= x_X = \frac{\rho_0}{\rho} \frac{v_0}{v} f, \\
\rho_0 \hat{e} &= \rho_0(X) \hat{e}(v_0, v, v_X, f, \eta) \equiv E(v, v_X, F; X), \quad (2.7)
\end{align}