Regularity Properties of Deformations with Finite Energy

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Abstract

The purpose of this paper is to point out some regularity properties of a class of functions which play an important rôle in nonlinear elasticity.

1. Introduction

Consider a material body occupying a region \( \Omega \subset \mathbb{R}^n \). Suppose the body is deformed and let us denote by \( \varphi(x) \) the position of the particle \( x \) after deformation. In the fundamental paper [1] J. Ball showed that the existence of solutions for a number of boundary-value problems in nonlinear elasticity can be obtained by establishing the existence of minimizers of the energy integral

\[
I(\varphi, \Omega) = \int_\Omega W(x, \varphi(x), \nabla \varphi(x)) \, dx
\]

and that, for \( n = 3 \), some appropriate classes of competing functions are contained in the classes

\[
\mathcal{A}_{p,q}(\Omega) = \{ \varphi : \Omega \mapsto \mathbb{R}^n, \nabla \varphi \in L^p(\Omega), \text{adj} \nabla \varphi \in L^q(\Omega) \}
\]

or, if we wish to be more realistic, in the classes

\[
\mathcal{A}_{p,q}^+(\Omega) = \{ \varphi : \Omega \mapsto \mathbb{R}^n, \varphi \in \mathcal{A}_{p,q}(\Omega), \text{det} \nabla \varphi > 0 \text{ a.e. in } \Omega \}
\]

where \( p \geq n - 1, \ q \geq \frac{p}{p-1} \).

These classes can be defined in the same way for any \( n \geq 2 \) and the purpose of this article is to point out some regularity properties of the functions contained in these classes.

It is of interest to know whether holes can form in the deformed body or, if we use the terminology introduced in Ball [3], whether cavitation can occur.
It turns out that for certain types of the stored energy functions $W$ cavitation does occur. This was proved in Ball [3] and the results of that paper were generalized in Stuart [1] and Sivaloganathan [1].

However, these functions do not satisfy the growth conditions assumed in the existence theory of Ball [1]. Here we show that for a class of stored energy functions considered in Ball [1] cavitation cannot occur. In fact we prove that if $\varphi \in \mathcal{A}_{p,q}(Q)$ with $p \geq n - 1$, $q \geq \frac{p}{p - 1}$, the boundary $\Gamma = \partial Q$ is smooth and the trace $\overline{\varphi} = \varphi | \Gamma$ is continuous and satisfies some other regularity conditions, then

$$\int_{Q} f(\varphi(x)) \det \nabla \varphi(x) \, dx = \int_{\mathbb{R}^n} f(y) \deg (\overline{\varphi}, Q, y) \, dy$$

for any bounded continuous function $f$. (See Corollary 1.) This formula clearly excludes cavitation by considering a function $f$ with support in the hole. Using (2) we derive some other regularity properties of the functions from the classes

$$\mathcal{A}_{p,q}^{+}(Q) \quad \text{with} \quad p > n - 1, \; q \geq \frac{p}{p - 1}.$$

We prove (see Theorem 4) that every function $\varphi \in \mathcal{A}_{p,q}^{+}(Q)$ with $n - 1 < p < n$, $q \geq \frac{p}{p - 1}$ has a representative $\overline{\varphi}$ which is continuous outside a set $S$ of Hausdorff dimension $n - p$. Moreover, it is possible to define in a natural way a set function $x \mapsto F(x, \varphi)$ such that $F(x, \varphi) = \{\overline{\varphi}(x)\}$ for each $x \in Q \setminus S$ and, for each $x \in S$, $F(x, \varphi)$ is a compact connected set describing the singularity of the deformation at $x$. We shall investigate some properties of this set function. (See Lemma 4 and Theorem 6.)

In the case $p = n$ it is possible to show the following. If $\varphi \in H_0^1(Q)$ and $\det \nabla \varphi > 0$ a.e. in $Q$, then $\varphi$ has a continuous representative. This was proved by Vodopyanov & Goldstein [1] and here the proof of this result appears naturally in Section 4.

In Section 5 we apply these results to a special situation similar to the one considered in Ball [2]. We shall suppose that $Q$ is smooth, $\varphi \in \mathcal{A}_{p,q}^{+}(Q)$, $p > n - 1$, $q \geq \frac{p}{p - 1}$ and that the deformation $\varphi$ satisfies $\varphi | \Gamma = \varphi_0 | \Gamma$ where $\varphi_0$ is a sufficiently regular homeomorphism of a neighborhood of $\overline{Q}$ into $\mathbb{R}^n$.

We show that we can define in a natural way an inverse function $\psi$ of the function $\varphi$. The function $\psi$ is continuous outside a set of $(n - 1)$-dimensional Hausdorff measure zero and always belongs to the space $H_1^0(\varphi_0(Q))$. Moreover, the usual formulae relating the derivatives of $\varphi$ and $\psi$ and expressing the $H_1^0$-norm of $\psi$ in terms of $\varphi$ are valid. (See Theorem 7 and Theorem 8.)

The proof of (2) is based on well-known formulae from the theory of the Brouwer degree. (See Section 2.) The proof of the assertions concerning the classes $\mathcal{A}_{p,q}^{+}$ is based on continuity properties of functions from $H_0^1$ with $p > n - 1$ on $(n - 1)$-dimensional submanifolds and on a monotony property of functions from $\mathcal{A}_{p,q}^{+}$. Roughly speaking, if $\varphi \in \mathcal{A}_{p,q}^{+}(Q)$, then each component of $\varphi$ is monotone.