Approximation-Solvability
of Nonlinear Functional Equations in Normed Linear Spaces

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Introduction

Let $X$ and $Y$ be two normed linear spaces, $T$ a (possibly) nonlinear mapping of $X$ into $Y$, and $f$ a given element of $Y$. By a nonlinear functional equation in the present context, we mean the problem of determining those elements $x$ in $X$ for which

$$(1) \quad T(x) = f.$$ 

We shall distinguish in principle between two concepts of solvability for such a problem, namely:

(I) Solvability, i.e., solvability in the purely existential sense that we can determine by some means that such elements $x$ actually exist in $X$;

(II) Approximation-solvability, i.e., the construction of a solution $x$ as the limit of a specified sequence of approximants which are solutions of simpler finite dimensional problems.

To make the latter concept more precise, we introduce the following two definitions:

**Definition 1.** By an approximation scheme for the mapping $T$ of $X$ into $Y$, we mean a quadruple of sequences

$$(X_n, Y_n, P_n, Q_n)$$

where for each positive integer $n$, $X_n$ is a finite dimensional subspace of $X$, $Y_n$ is a finite dimensional subspace of $Y$, $P_n$ is a mapping of $X$ into $X_n$, $Q_n$ is a continuous mapping of $Y$ into $Y_n$, and the following conditions hold:

1. For each $n$, the dimension of $X_n$ equals the dimension of $Y_n$.
2. For each $x$ in $X$, $P_n x$ converges to $x$ strongly in $X$ as $n \to + \infty$.
3. For each $y$ in $Y$, $Q_n y$ converges to $y$ strongly in $Y$ as $n \to + \infty$, uniformly on compact subsets of $Y$.

**Definition 2.** Given a mapping $T$ of $X$ into $Y$, an element $f$ of $Y$, and an approximation-scheme in the sense of Definition 1, we shall say that the equation $T(x) = f$ is approximation-solvable with respect to the given scheme if both the following conditions are satisfied:

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(a) If we set $T_n = Q_n T|_{X_n}$ for each $n$ there exists at least one solution $x_n$ in $X_n$ of the $n$-th approximating equation

$$T_n(x_n) = Q_n(f), \quad (x_n \in X_n).$$

(b) The approximants $x_n$ converge strongly in $X$ to a solution $x$ of the equation $T(x) = f$.

We shall say that the equation $T(x) = f$ is uniquely approximation-solvable if both the approximants $x_n$ for each $n$ and the limit solution $x$ are unique.

It follows obviously from condition (b) of Definition 2 that every equation $T(x) = f$ which is approximation-solvable with respect to a given scheme is also solvable in the ordinary sense. We remark that if $X_n = Y_n$ for each $n$ and if $P_n = Q_n$ is a linear projection mapping, Definition 2 essentially coincides with the definition given by PETRYSYN [23] (at least for unique approximation-solvability) of projectional solvability in the unique and strong sense.

It has been frequently and correctly argued (e.g., as in [23]) that the distinction between solvability in the general sense (I) and approximation-solvability in the sense (II), as specified for example by Definitions 1 and 2, corresponds rather sharply to the distinction between abstract functional analysis on the one hand and numerical (or constructive) functional analysis on the other. From the point of view of numerical functional analysis (and more generally of the applied mathematical purposes which it designs to serve) it is important not just to know of the abstract existence of solutions of functional equations but much more essentially to construct methods of calculating such solutions (preferably by iterative methods or by finite-dimensional approximations of some sort).

It is our purpose in the present paper to put forward a sharp antithesis to this argument, namely: It is true under fairly general circumstances that the knowledge of the existence of a solution of the equation $T(x) = f$ can be an essential tool in proving that the equation $T(x) = f$ is approximation-solvable by a given scheme, no matter how the knowledge of the existence of solutions is obtained.

In a previous note [12], we have formulated a principle of this sort in a narrower context and used it to obtain the convergence of Galerkin approximations for functional equations involving nonlinear accretive (i.e., $J$-monotone) operators in rather general Banach spaces. The approximation result was made to rest thereby upon the existence theory for this class of operators, which is (cf. [11]) basically of a constructive character.

In the present paper, we formulate much more general results by which one can pass in a relatively general context from solvability to approximation-solvability, and we use these general theorems to establish the convergence to solutions of Galerkin approximations for nonlinear functional equations involving complex monotone (as well as monotone) operators from a complex Banach space $X$ to its conjugate space $X^*$. In the latter context, the existence theorems which we give below (and which are refinements of preceding work in this direction) are basically non-constructive. In the present case, therefore, the convergence proof of the standard type of sequential approximation to the unique solution of the given equation can be carried through (at least by the methods available at present) only