The Role of Inertia in the Hydrodynamics of Porous Media

P. A. C. RAATS

Communicated by C. TRUESDELL

Introduction

It is widely believed that one may account for the role of inertia in the hydrodynamics of porous media by simply introducing an inertial force in the differential balance of forces. However, MÜLLER [1971, 1] has recently pointed out that inertia should also enter into the jump conditions for the forces. In Chapter I, I formulate the differential equations and jump conditions describing the flow of a homogenous, incompressible, ideal fluid in a homogeneous, rigid porous medium. In Chapter II, I analyze the rotationality associated with the inertial force. In Chapter III, I discuss some special properties of irrotational flows. In Chapter IV, I present a detailed discussion of a steady, plane rotational flow near an interface and evaluate the boundary conditions that are traditionally used.

I. Preliminaries

1. Some Concepts from Kinematics. Consider two bodies \( B_\alpha \), \( \alpha = 1, 2 \), which occupy regions of space, possibly simultaneously [1969, 1, section 5]. Let \( X_a \) be the place occupied by a particle of \( B_\alpha \) in some reference configuration. The motion of \( B_\alpha \) is the mapping

\[
x = x_a(X_a, t), \quad -\infty < t < \infty,
\]

of \( B_\alpha \) onto a time-sequence of configurations in space. Let a backward prime denote the time derivative when \( X_a \) is held constant, i.e., a time derivative that is material with respect to \( B_\alpha \). The velocity \( v_a \) and the acceleration \( b_a \) are defined as

\[
v_a \equiv \dot{x}_a = \frac{\partial}{\partial t} x_a(X_a, t),
\]

\[
b_a \equiv \ddot{x}_a = \frac{\partial^2}{\partial t^2} x_a(X_a, t).
\]

The deformation gradient \( F_a \) is defined as [1960, 2, section 17]*

\[
F_a \equiv \frac{\partial}{\partial X_a} x_a(X_a, t).
\]

* Many of the basic concepts used in this study are also discussed in The Kinematics of Vorticity by TRUESDELL [1954, 1].
The vorticity vector $\mathbf{w}_a$ is defined as [1960, 2, section 86]

$$\mathbf{w}_a \equiv \text{curl } \mathbf{v}_a.$$  \hspace{1cm} (1.5)

The second vorticity vector $\mathbf{w}^*_a$ is defined as [1960, 2, section 107; 1968, 1]

$$\mathbf{w}^*_a \equiv \text{curl } \mathbf{b}_a.$$  \hspace{1cm} (1.6)

The circulation of a vector, say $\mathbf{v}_a$, around a material curve $C_a$ is defined as [1960, 2, section 87]

$$\oint_{C_a} \mathbf{v}_a \cdot d\mathbf{x}.$$  \hspace{1cm} (1.7)

KELVIN'S transformation states that the circulation of $\mathbf{v}_a$ around $C_a$ is equal to the flux of vorticity $\mathbf{w}_a$ through any surface $S$ bounded by $C_a$ [1960, 1, section 28]:

$$\oint_{C_a} \mathbf{v}_a \cdot d\mathbf{x} = \iint_S \mathbf{w}_a \cdot d\mathbf{a}.$$  \hspace{1cm} (1.8)

Let $\Psi$ be a field that is continuously differentiable in some region, except at some singular surface $s$. Let $x = x(l)$ be a smooth curve upon $s$. HADAMARD’S condition of compatibility states that the jump of the tangential derivative of $\Psi$ is equal to the tangential derivative of the jump of $\Psi$ [1960, 2, section 175]:

$$\frac{d}{dl} \left[ \Psi \right] = \left[ \frac{\partial_k \Psi}{} \right] \frac{dx_k}{dl},$$  \hspace{1cm} (1.9)

where $[\cdot]$ denotes the jump of a quantity at a singular surface. MAXWELL’S theorem states that if $[\Psi] = 0$, or if $\Psi$ is a scalar and $[\Psi] = \text{constant}$, then [1960, 2, section 175]

$$[\text{grad } \Psi] = A \mathbf{n}, \quad A \equiv \left[ n \cdot \text{grad } \Psi \right] = \frac{\partial \Psi}{\partial n},$$ \hspace{1cm} (1.10)

i.e., then $[\text{grad } \Psi]$ is normal to the singular surface.

2. The General Balance Equation. Consider a material volume $V_a$ with a surface $S_a$ [1960, 2, section 192]. Let $s$ be a smooth surface which divides $V_a$ into two parts, $V_a^+$ and $V_a^-$, and $S_a$ into two parts, $S_a^+$ and $S_a^-$. Let $u_n$ be the speed of displacement of $s$. The surface $s$ is assumed to be singular with respect to some quantity $\psi_a$ and possibly also with respect to $v_a$. The general balance equation for a quantity $\psi_a$ in $V_a$ may be written as [1960, 2, section 157]

$$\left( \iint_{V_a} \rho_a \psi_a \, dv \right) = - \iint_{S_a} i_a \cdot d\mathbf{a} + \iint_{V_a} \rho_a s_a \, dv,$$  \hspace{1cm} (2.1)

where $\rho_a$ is the mass density of $V_a$, $i_a$ is the influx of $\psi_a$ through $S_a$, and $s_a$ is the supply of $\psi_a$ within $V_a$. The left hand side of (2.1) may be expanded into three terms [1960, 2, sections 192–193]:

$$\iint_{V_a} \left[ \frac{\partial \rho_a \psi_a}{\partial t} \right] \, dv + \iint_{S_a} \rho_a \psi_a v_{an} \, d\mathbf{a} - \iint_{S_a} [\rho_a \psi_a] u_n \, d\mathbf{a} = - \iint_{S_a} i_a \cdot d\mathbf{a} + \iint_{V_a} \rho_a s_a \, dv,$$  \hspace{1cm} (2.2)

where the subscript $n$ denotes a component normal to a surface.