Extremal Problems for a Class of Symmetric Functions

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Communicated by M. M. SCHIFFER

1. Introduction

Let $R$ denote the annulus \( \{ z : r_0 < |z| < 1 \} \). D. GAIER [3] introduced the family $\mathcal{F}$ of functions $f(z)$ that are holomorphic and schlicht in $R$ and satisfy the three conditions

(1) \[ |f(z)| < 1 \quad (z \in R), \quad |f(z)| = 1 \quad (|z| = 1), \]

(2) \[ f(z) \neq 0 \quad (z \in R), \]

(3) \[ f(1) = 1. \]

Since then, many extremal problems for the class $\mathcal{F}$ and related classes have been considered ([1], [2], [4] to [9]).

In many of these extremal problems, the extremal function is symmetric with respect to the real axis; or if an extremal function is not unique, there exists a symmetric extremal function [7]. This leads us to consider the compact subclass $\mathcal{F}_s$ of functions whose image is symmetric with respect to the real axis:

$$\mathcal{F}_s = \{ f : f \in \mathcal{F}, f(z) = f(\bar{z}) \}.$$

We shall develop a variational formula for $\mathcal{F}_s$ and use it to solve extremal problems in $\mathcal{F}_s$. In most cases, we shall obtain unique extremal functions. Several extremal problems that appear to be inaccessible in $\mathcal{F}$ (because many parameters are involved) can easily be solved in the subclass $\mathcal{F}_s$.

Of course, if an extremal function for $\mathcal{F}$ belongs to $\mathcal{F}_s$, it is also extremal in the smaller class. Thus, in Sections 3 and 4, our method yields known results.

2. A Variational Formula for $\mathcal{F}_s$

Each $f \in \mathcal{F}$ maps $R$ onto the unit disk minus some continuum $\Gamma_f$ containing the origin. If $f \in \mathcal{F}_s$, then $\Gamma_f$ is symmetric with respect to the real axis.

Now let $f$ belong to $\mathcal{F}_s$. Fix $w_0 \in \Gamma_f$, say $w_0 \neq 0$. Let $D_\rho(w_0)$ ($\rho > 0$) denote the domain consisting of all points either exterior to $\Gamma_f$ or exterior to the disk $|w - w_0| \leq \rho$. It is known [2], [10] that there exist functions of the form

$$F(w) = w + \frac{a \rho^2 w}{(w - w_0) w_0} + O(\rho^3)$$
that are analytic and univalent in \( D_\rho(w_0) \) and that leave the origin fixed. Here the constant \( a \) depends on \( \rho \) and \( |a|=|a(\rho)|\leq 1 \), and the error term \( O(\rho^2) \) can be estimated uniformly in each closed subdomain of \( D_\rho(w_0) \).

If \( w_0 \) is real, it follows [10] that the function \( F(w) \) can be chosen to have real coefficients, so that \( F(w)=\overline{F(w)} \). Now suppose \( w_0 \) is not real, say \( \text{Im} \ w_0 > 0 \). Choose \( \rho \) so small that \( |w-w_0|=\rho \) does not intersect the real axis. There exists a function \( h(w)=w/(w-\overline{w_0})+O(\rho) \), defined and satisfying a Lipschitz condition in the half-plane \( \text{Im} \ w>\text{Im} \ \overline{w_0}/2 \) (this implies that for each constant \( c \) the function \( w+c\rho^2h(w) \) is univalent for sufficiently small \( \rho \)), such that

\[
H(w)=F(w)+\frac{\bar{a}\rho^2}{w_0} h(F(w))
\]

is univalent in \( D_\rho(w_0) \cap \{w: \text{Im} \ w\geq 0\} \) and maps the real axis onto the real axis with \( H(0)=0 \), for all sufficiently small \( \rho \). A computation shows that

\[
H(w)=w+\frac{a\rho^2w}{w-w_0} \frac{\bar{a}\rho^2w}{w-\overline{w_0}} + O(\rho^3).
\]

Now extend \( H \) to a univalent function in \( D_\rho(w_0) \cap D_\rho(\overline{w_0}) \) by setting \( H(w)=\overline{H(w)} \). (Clearly, if \( w_0 \) is real, then \( a \) can be chosen to be real, and \( H \) has the same form as \( F \).)

Proceeding as in [2], computing first the function

\[
W=\frac{a\rho^2W^2}{1-w_0 W} - \frac{\bar{a}\rho^2W^2}{1-\overline{w_0} W}, \quad W=H(w),
\]

we find the following variational formula for the class \( \mathcal{F}_1 \):

\[
V_\rho^2(w)=w \left[ 1 + a\rho^2 \frac{1-w^2}{w_0(w-w_0)(1-w_0 w)} + \bar{a}\rho^2 \frac{1-w^2}{\overline{w_0}(w-\overline{w_0})(1-\overline{w_0} w)} \right] + O(\rho^3).
\]

3. Maximum and Minimum Value of \( |f(z)| \)

Suppose that \( f \) is an extremal function for the problem

\[
\max_{g \in \mathcal{F}} |g(z)|.
\]

Then \( |V_\rho^2(f(z))| \leq |f(z)| \). Set \( \alpha=f(z) \). The last inequality leads to the relation

\[
\text{Re} \left\{ a\rho^2 \left[ \frac{1-\alpha^2}{w_0(\alpha-w_0)(1-\alpha w_0)} + \frac{1-\overline{\alpha}^2}{\overline{w_0}(\overline{\alpha}-\overline{w_0})(1-\overline{\alpha} w_0)} \right] + O(\rho^3) \right\} \leq 0.
\]

It now follows from SCHIFFER'S lemma [10] that the continuum \( \Gamma_f \) satisfies the differential equation \( w'(t)^2 s(w(t))>0 \), where

\[
s(w)=\frac{\text{Re} \alpha-w(1+|\alpha|^2)+w^2 \text{Re} \alpha}{w(\alpha-w)(1-\alpha w)(\overline{\alpha}-w)(1-\overline{\alpha} w)}.
\]