Extremal Problems for a Class of Symmetric Functions

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1. Introduction

Let \( R \) denote the annulus \( \{ z : r_0 < |z| < 1 \} \). D. GAIER [3] introduced the family \( \mathcal{F} \) of functions \( f(z) \) that are holomorphic and schlicht in \( R \) and satisfy the three conditions

\begin{align*}
(1) & \quad |f(z)| < 1 \quad (z \in R), \quad |f(z)| = 1 \quad (|z| = 1), \\
(2) & \quad f(z) \neq 0 \quad (z \in R), \\
(3) & \quad f(1) = 1.
\end{align*}

Since then, many extremal problems for the class \( \mathcal{F} \) and related classes have been considered ([1], [2], [4] to [9]).

In many of these extremal problems, the extremal function is symmetric with respect to the real axis; or if an extremal function is not unique, there exists a symmetric extremal function [7]. This leads us to consider the compact subclass \( \mathcal{F}_s \) of functions whose image is symmetric with respect to the real axis:

\[ \mathcal{F}_s = \{ f : f \in \mathcal{F}, f(z) = f(\bar{z}) \}. \]

We shall develop a variational formula for \( \mathcal{F}_s \) and use it to solve extremal problems in \( \mathcal{F}_s \). In most cases, we shall obtain unique extremal functions. Several extremal problems that appear to be inaccessible in \( \mathcal{F} \) (because many parameters are involved) can easily be solved in the subclass \( \mathcal{F}_s \).

Of course, if an extremal function for \( \mathcal{F} \) belongs to \( \mathcal{F}_s \), it is also extremal in the smaller class. Thus, in Sections 3 and 4, our method yields known results.

2. A Variational Formula for \( \mathcal{F}_s \)

Each \( f \in \mathcal{F} \) maps \( R \) onto the unit disk minus some continuum \( \Gamma_f \) containing the origin. If \( f \in \mathcal{F}_s \), then \( \Gamma_f \) is symmetric with respect to the real axis.

Now let \( f \) belong to \( \mathcal{F}_s \). Fix \( w_0 \in \Gamma_f \), say \( w_0 \neq 0 \). Let \( D_\rho(w_0) (\rho > 0) \) denote the domain consisting of all points either exterior to \( \Gamma_f \) or exterior to the disk \( |w - w_0| \leq \rho \). It is known [2], [10] that there exist functions of the form

\[ F(w) = w + \frac{a \rho^2 w}{(w - w_0) w_0} + O(\rho^3) \]
that are analytic and univalent in $D_{\rho}(w_0)$ and that leave the origin fixed. Here the constant $a$ depends on $\rho$ and $|a|=|a(\rho)| \leq 1$, and the error term $O(\rho^3)$ can be estimated uniformly in each closed subdomain of $D_{\rho}(w_0)$.

If $w_0$ is real, it follows [10] that the function $F(w)$ can be chosen to have real coefficients, so that $F(w)=\overline{F(w)}$. Now suppose $w_0$ is not real, say $\text{Im } w_0 > 0$. Choose $\rho$ so small that $|w-w_0|=\rho$ does not intersect the real axis. There exists a function $h(w)=w/(w-w_0)+O(\rho)$, defined and satisfying a Lipschitz condition in the half-plane $\text{Im } w > \text{Im } w_0/2$ (this implies that for each constant $c$ the function $w+c\rho^2 h(w)$ is univalent for sufficiently small $\rho$), such that

$$H(w)=F(w)+\frac{\bar{a}\rho^2}{w_0}h(F(w))$$

is univalent in $D_{\rho}(w_0) \cap \{w: \text{Im } w \geq 0\}$ and maps the real axis onto the real axis with $H(0)=0$, for all sufficiently small $\rho$. A computation shows that

$$H(w)=w+\frac{a\rho^2 w}{(w-w_0)w_0}+\frac{\bar{a}\rho^2 w}{(w-w_0)w_0}+O(\rho^3).$$

Now extend $H$ to a univalent function in $D_{\rho}(w_0) \cap D_{\rho}(\bar{w}_0)$ by setting $H(w)=\overline{H(w)}$. (Clearly, if $w_0$ is real, then $a$ can be chosen to be real, and $H$ has the same form as $F$.)

Proceeding as in [2], computing first the function

$$W=\frac{a\rho^2 W^2}{(1-w_0 W)w_0}+\frac{\bar{a}\rho^2 W^2}{(1-w_0 W)w_0}, \quad W=H(w),$$

we find the following variational formula for the class $\mathcal{F}$:

$$V_{\rho}^2(w)=w\left[1+a\rho^2 \frac{1-w^2}{w_0(w-w_0)(1-w_0 w)}+\bar{a}\rho^2 \frac{1-w^2}{w_0(w-w_0)(1-w_0 w)}\right]+O(\rho^3).$$

### 3. Maximum and Minimum Value of $|f(z)|$

Suppose that $f$ is an extremal function for the problem

$$\max_{g \in \mathcal{F}} |g(z)|.$$

Then $|V_{\rho}^2(f(z))| \leq |f(z)|$. Set $\alpha=f(z)$. The last inequality leads to the relation

$$\Re \left\{ a\rho^2 \left[ \frac{1-\alpha^2}{w_0(\alpha-w_0)(1-\alpha w_0)} + \frac{1-\bar{\alpha}^2}{w_0(\bar{\alpha}-w_0)(1-\bar{\alpha} w_0)} \right] + O(\rho^3) \right\} \leq 0.$$

It now follows from SCHIFFER’s lemma [10] that the continuum $\Gamma_f$ satisfies the differential equation $w'(t)^2 s(w(t))>0$, where

$$s(w)=\frac{\Re \alpha-w(1+|\alpha|^2)+w^2 \Re \alpha}{w(\alpha-w)(1-\alpha w)(1-\bar{\alpha} w)}.$$