On the Entropy Inequality

INGO MÜLLER

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1. Introduction

The Thermodynamics of Irreversible Processes as a phenomenological theory
describing processes in continua was initiated by ECKART [1] in 1940. Independently
of ECKART's work, MEIXNER proposed essentially the same theory in a series of
papers between 1939 and 1943*. Both authors introduce an equation of balance
of entropy with positive production density. An important feature of this balance
equation is that the entropy flux is assumed to be equal to the heat flux divided
by the temperature, although this relation does not result from the theory; one
can suggest possibly meaningful generalizations of this assumption [3].

The motivation for this relation rests upon the definition of entropy in thermo-
statics and on an approximate calculation of the entropy flux based on the kinetic
type of gases.

In recent years, COLEMAN & NOLL [4] have developed an improved method
for exploiting the entropy balance. This method was applied to simple materials
with fading memory by COLEMAN [5]. Here again the postulate is made that
entropy flux and heat flux over temperature are equal.

In the present paper this assumption is omitted. Instead, we introduce an
independent entropy flux, subject to constitutive assumptions like those made for
heat flux, internal energy, stress, and entropy. By evaluation of the entropy in-
equality and application of a natural invariance principle, we are then able to
derive a relation between entropy flux and heat flux which, for simple materials
with fading memory, reduces to that usually postulated, except if these materials
have uncommon symmetries. Calculations for a dipolar fluid, however, seem to
indicate that the generalization of the entropy flux leads to alterations in the theory
of multipolar materials.

* See the survey by J. MEIXNER & H. G. REIK [2].
In a recent publication Gurtin & Williams [6] have generalized the entropy balance in a different way, but in their work also the entropy flux is proportional to heat flux. However the kinetic theory of gases gives a motivation for the assumption of a more general entropy flux [3].

2. Basic Concepts*

We consider a body $\mathcal{B}$, whose particles are characterised by the material co-ordinates $X_A$. We take the $X_A$ as coordinates of positions occupied by the particles in a reference configuration.

The motion of the body is then described by the function $x_i(X_A, t)$, which gives the position of the particles at time $t$. We call the function $x_i(X_A, t)$ the deformation and suppose that the deformation gradient

$$F_{iA}(X_B, t) = \frac{\partial x_i}{\partial X_A} \quad (2.1)$$

is nonsingular, i.e.

$$J = \det \{F_{iA}\} \neq 0. \quad (2.2)$$

Without loss of generality we may then assume: $J > 0$. The mass density $\rho$ is given by

$$\rho(X_A, t) = \frac{1}{J} \rho_0(X_A), \quad (2.3)$$

where $\rho_0(X_A)$ is the mass density in the reference configuration. The deformation gradient may be expressed as the product

$$F_{jB} = R_{jk} U_{kB}, \quad (2.4)$$

where $R_{jk}$ and $U_{kB}$ are components of a proper orthogonal tensor and a symmetric positive-definite tensor, respectively.

We suppose that it is always possible to assign a positive temperature $\theta(X_A, t)$ to each $X_A \in \mathcal{B}$.

For any deformation of the body, the equations of balance of linear momentum, moment of momentum, and internal energy hold. Hence

$$\rho \ddot{x}_i - \frac{\partial t_{ij}}{\partial x_j} - \rho b_i = 0, \quad (2.5)$$

$$t_{ij} = t_{ji}, \quad (2.6)$$

$$\rho \dot{\varepsilon} + \frac{\partial q_i}{\partial x_i} - t_{ij} \frac{\partial x_i}{\partial x_j} - \rho r = 0, \quad (2.7)$$

where $t_{ij}$ is the stress tensor, $b_i$ the specific body force, $\varepsilon$ the specific internal energy, $q_i$ the heat flux vector, and $r$ the specific energy supply from the external world, per unit time. The dot denotes the material time derivative.

* Throughout this paper we employ Cartesian tensor notation.