Spatial and spatio-temporal patterns in a cell-haptotaxis model

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Abstract. We investigate a cell-haptotaxis model for the generation of spatial and spatio-temporal patterns in one dimension. We analyse the steady state problem for specific boundary conditions and show the existence of spatially heterogeneous steady states. A linear analysis shows that stability is lost through a Hopf bifurcation. We carry out a nonlinear multi-time scale perturbation procedure to study the evolution of the resulting spatio-temporal patterns. We also analyse the model in a parameter domain wherein it exhibits a singular dispersion relation.

Key words: Morphogenesis — Spatio-temporal pattern — Mechanochemical — Haptotaxis — Singular dispersion relation

1. Introduction

Several models have been proposed to describe cell pattern formation in early embryonic development. A large number of these models are based on reaction diffusion systems involving chemicals which, in the appropriate parameter domain, bifurcate from a homogeneous steady state and evolve into a heterogeneous steady state (see, for example, [1–3]). This chemical pre-pattern is then interpreted by cells which differentiate accordingly to form a cell pattern [4].

Recently, an alternative approach to embryological pattern formation has been proposed by Murray, Oster and their co-workers [5, 6] based on the mechanochemical interaction of cells with their surrounding tissue. Linear analysis of such models suggests that they exhibit wide-ranging pattern formation capability [7]. Perelson et al. [8] confirmed this by numerically solving the nonlinear system in one dimension to get spatially heterogeneous and spatio-temporally oscillating patterns. Maini and Murray [9] carried out a nonlinear analysis for a simple version of the model to investigate spatial pattern formation.
In this paper we analyse a more complicated version of the model to determine possible spatial patterns and spatio-temporally oscillating solutions. In Sect. 2 we briefly motivate the model equations. In Sect. 3 we investigate the linear stability of the non-trivial steady state and show that it is unstable to perturbations in a certain range of wave numbers. In Sect. 4 we investigate the possible spatial patterns the model can exhibit. In Sect. 5 we consider periodic boundary conditions. Under these conditions the spatially uniform steady state loses stability through a Hopf bifurcation and we examine the evolution of such spatio-temporally oscillating solutions using the method of multi-time scale perturbation (see, for example, [10]). In Sect. 6, we investigate a singular dispersion relation that arises in a certain parameter domain for the model. We analyse the subsequent fast-focussing problem and show the possible existence of small amplitude spatial pattern.

2. The model

In 1983, Oster et al. [5] proposed a mechanical model for pattern formation based on mesenchymal cells moving through and deforming an extracellular matrix (ECM). In this section we briefly describe a simplified version of the full model and refer the reader to the original paper for full details. The model is based on the three field variables:

\[ n(x, t) = \text{density of mesenchymal cells at position } x \text{ and time } t, \]
\[ \rho(x, t) = \text{density of ECM at position } x \text{ and time } t, \]
\[ u(x, t) = \text{displacement at time } t \text{ of a material point of ECM initially at } x. \]

The model consists of three equations:

Cell conservation.

\[ \frac{\partial n}{\partial t} = -\nabla \cdot J \]

where \( J \) is a flux term. Several cell processes contribute to this term. We shall focus on three in particular, namely, diffusion, convection and haptotaxis.

The diffusive flux is of Fickian type,

\[ J_D = -D \nabla n \]

where \( D \) is the positive diffusion coefficient.

The convective flux is due to cells riding passively on the ECM and may be modeled by

\[ J_C = n \frac{\partial u}{\partial t}. \]

Cells move by attaching their filopodia to specialized adhesive sites in the ECM and move up a gradient in adhesive site density. This process is called haptotaxis.