Simulation of Rhythmic Nervous Activities

II. Mathematical Models for the Function of Networks with Cyclic Inhibition

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Zusammenfassung. Die in diesem Artikel beschriebenen mathematischen Modelle geben eine allgemeine exakte Behandlung des Verhaltens einfacher Netzwerke, die in den Abschnitten 3.1.1—3.1.8 des Teiles I eingeführt wurden. Gerichtete Graphen werden studiert, deren Kanten hemmend wirken und deren Punkte durch eine ständige „Hintergrundwirkung“ angeregt sind, wobei die Stimulation einen linearen Anstieg des Erregungswertes hervorruft. Die Graphen enthalten keine Schlingen und kein Paar übereinstimmender gerichteter paralleler Kant. Eine weitere Beschränkung der Struktur der Graphen wird nicht gefordert. Im Gegenteil, es ist eine wesentliche Annahme, daß der Reiz auf jeden Punkt in genau der selben Weise wirkt. — Das im Abschnitt 3 konzipierte Modell beschreibt, wie ein Graph von einem gegebenen Anfangszustand aus funktioniert. Die Zeitpunkte \( T_i \), \( i = 1, \ldots \), die den Beginn jeder oder mehrerer Hemmungsphasen bezeichnen, heißen Sprungmomente. Sind die Erregungswerte im Sprungmoment \( T_j \) bekannt, so definieren wir das folgende Sprungmoment \( T_{j+1} \), ferner bestimmen wir durch die Formeln (6.1) die Erregungswerte, die den Punkt zwischen den Zeitpunkten \( T_j \) und \( T_{j+1} \) zugeordnet werden. — Im Abschnitt 4 wird ein anderes mathematisches Modell eingeführt. Die virtuellen Sprungmomente \( T_m \), \( m = 1, \ldots \), werden in der Formel (4.1) definiert, die Matrix (4.2) enthält alle die zu diesen Zeitpunkten möglichen Werte. Die Formeln (4.6)—(4.10) beschreiben, wie die Mengen der Punkte, die die verschiedenen Werte haben, von \( T_m \) zu \( T_{m+1} \) sich verändern. Feststellung 8 behauptet, daß das im Abschnitt 4 enthaltene Modell die wesentlichen Aspekte des Vorganges ausdrückt, der durch das im Abschnitt 3 ausgearbeitete Modell formuliert worden ist.

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Note that \( \tau \) corresponds to the quantity \( R \) (the length of the recovery phase) in Part I.
We accept the hypothesis that the inhibition acts instantly. Under certain circumstances, this assumption can lead to ambiguity. For example, let us consider the situation when

(i) there are three vertices P, Q, R connected cyclically in the graph (i.e. there exist three edges from P towards Q, from Q towards R and from R towards P),
(ii) at some instant t, the vertices P, Q, R have the same value \( x(0 \leq x < 1) \), and
(iii) when considering their behaviour, none of the vertices P, Q, R is inhibited by a fourth vertex.

If (i), (ii), (iii) are fulfilled, then the values of P, Q, R grow linearly and simultaneously. The continuous extension of this growth shows that each of the three values reaches 1 in the same instant \( t_0 \). However, all of these values must be 0 by their inhibitory interaction at this instant. Ambiguity occurs in that we have no plausible reason to decide which of the vertices P, Q, R should be inhibiting and which of them should be 0 (in consequence of inhibition).

Because of the possibility of similar inconsistencies arising, we shall exclude the situation, concerning the instants \( \leq t_0 \), when two vertices P, Q are connected by an edge and they simultaneously converge to the value 1 at an instant \( t_0 \).

The present paper deals with two mathematical models formalizing the process described above. These models differ from each other in that the model described in Sect. 3 has a continuous scale of time and values, and the "essential" instants (i.e. the instants when inhibitions really appear) are determined step by step. In the discrete model (Sect. 4), the scales of the (potentially) essential instants and possible values are predetermined. The continuous model is nearer to the physical process, the discrete one gives a more comprehensive insight into its essential aspects.

2. Preliminaries on Sets, Numbers and Graphs

We suppose that the reader has a certain familiarity with mathematics. The subsequent presentation of some mathematical concepts and notations is a reminder rather than an exhaustive exposition.

The set-theoretical signs \( \subseteq, \subseteq^*, \cap, \cap^* \), \( \cap, \cap^* \), \( \cup \) are assumed to be known. The sign \(-\), applied for sets, denotes the set-theoretical difference. The operations \( \cup, \cap \) are commutative and associative. Moreover, the following identities hold:

\[
(A \cup B) \cap C = (A \cap C) \cup (B \cap C), \\
(A \cap B) \cup C = (A \cup C) \cap (B \cup C), \\
A - B = A -(A \cap B), \\
(A - B) - C = A -(B \cup C), \\
(A - B) \cap C = (A \cap C) - B, \\
(A - B) \cup B = A \cup B, \\
(A \cap (B \cup C)) - B = (A \cap C) - B.
\]

Let \( a, b \) be two real numbers such that \( a \leq b \). We denote by \([a, b]\) the closed interval between \( a \) and \( b \), i.e. the set of the real numbers \( x \) which satisfy \( a \leq x \leq b \).

Similarly, the open interval, determined by \( a < x < b \), is denoted by \((a, b)\). The meaning of the notation \([a, b]\) and \((a, b)\) is also clear.

For any real number \( a \), we denote by \([a]\) the largest integer which does not exceed \( a \), and by \([a]\) the difference \( a - [a] \). We have \( 0 \leq [a] < 1 \) for any \( a \). Let \( a, b \) be two integers; the quotient and the remainder of the arithmetical division of \( a \) by \( b \) are evidently \([a/b]\) and \( b \cdot [a/b] \), respectively.

A (directed) graph is a mathematical structure consisting of a finite number of vertices and a finite number of edges. To any edge \( e \) there exists exactly one initial vertex \( P \) and exactly one final vertex \( Q \) of \( e \), where \( P \) and \( Q \) are different. We assume that for any ordered pair \( P, Q \) there exists at most one edge from \( P \) towards \( Q \).

We denote by \( \Psi \) the set of all vertices of the graph. If \( P \) is a vertex of a graph, we denote the set of the final vertices \( Q \) receiving edges from \( P \) by \( \tau(P) \)- More generally, let \( \Psi' \) be a subset of the set \( \Psi \); we denote by \( \chi(\Psi') \) the union of the sets \( \chi(P) \) where \( P \) runs through the elements of \( \Psi' \). The following relations hold:

\[
\chi(\Psi' \cup \Psi'') = \chi(\Psi') \cup \chi(\Psi''), \\
\chi(\Psi' \cap \Psi'') \neq \chi(\Psi') \cap \chi(\Psi''), \\
\text{if } \Psi' \subseteq \Psi'', \text{ then } \chi(\Psi'') \subseteq \chi(\Psi')
\]

where \( \Psi', \Psi'' \) are arbitrary subsets of \( \Psi \).

3. The Continuous Model

Let a graph and a positive real number \( \tau \) be given. Denote the vertices of the graph by \( P_1, P_2, \ldots, P_n \), the set of these vertices of the graph by \( \Psi \). Let the real numbers \( \alpha_1(0), \alpha_2(0), \ldots, \alpha_n(0) \) be assigned to the vertices, resp.; assume that each of these numbers lies in the interval \([0, 1]\). The corresponding vertices and numbers are labelled by identical indices. The vector

\[
\mathbf{\varphi} = \langle \alpha_1(0), \alpha_2(0), \ldots, \alpha_n(0) \rangle
\]

is called an initial state. Suppose this state satisfies the requirement: if \( P_i \in \chi(P_j) \) and \( \alpha_i(0) = 1 \), then \( \alpha_i(0) = 0 \).

Starting with the notations introduced above, we shall define \( n \) numerical functions \( \alpha_1(t), \alpha_2(t), \ldots, \alpha_n(t) \) such that the variable \( t \) (interpreted as time) runs through real numbers forming an interval \([0, T_{\text{max}}) \) or \([0, \infty) \), and the values of these functions are real numbers lying in the interval \([0, 1]\). The collection of these functions is called the activity of the graph. This activity depends (i) on the structure of the graph, (ii) on the time span \( \tau \) during which the state of the vertices changes from 0 to 1, and (iii) on the initial state \( \langle \alpha_1(0), \alpha_2(0), \ldots, \alpha_n(0) \rangle \) formed by the actual value of the vertices at the onset of the activity. We note that \( \tau \) is common for the vertices, the same holds for the (possibly existing) upper bound \( \tau_{\text{max}} \).

We shall determine the functions \( \alpha_i(t) \) \((i = 1, 2, \ldots, n)\) successively in certain time intervals \((T_k, T_{k+1}] \) surrounded by the spring instants \( T_k \) and \( T_{k+1} \) successively in certain time intervals \((T_k, T_{k+1}] \) surrounded by the spring instants \( T_k \) and \( T_{k+1} \).

Let \( T_0' \) be 0. Assume that the functions \( \alpha_i(t) \) are defined if \( T_k \leq t \leq T_k' \) (where \( k \) can be 0, 1, 2, \ldots).

Denote by \( \delta_k \) the greatest of the real numbers

\[
\{\alpha_1(T_0'), \alpha_2(T_0'), \ldots, \alpha_n(T_0')\}
\]

In the sense of the notation introduced in Sect. 2, \( \{z\} \) means the "fraction part" of the number \( z \). Since, in the present case, \( \alpha_i(T_0') \) varies from 0 to 1, \( \alpha_i(T_0') \) \( \{z\} \) Consequently, any two vertices can be joined by at most two edges; if two such edges do exist, they are directed in opposite manner.

We write a prime to the letter \( T \) in order to distinguish these instants from the time scale to be introduced in Sect. 4.