AN EXAMPLE OF THE PASSAGE FROM THE CONCRETE TO THE MANIPULATION OF FORMAL SYSTEMS

INTRODUCTION

Everybody knows that mathematics is an abstract subject. It follows that most people who have studied the problem of learning such an abstract subject, would agree that some passage from the concrete to the abstract must be mapped. By concrete, we mean usually, our immediate contact with the real world. We come into contact with objects and events and we re-act to them. This is the concrete level at which all organisms behave until they are able to organize their re-actions to events into re-actions to sets of events. This is the first stage towards abstraction – when the organism begins to classify. It is of interest to try and investigate the details of the abstraction process, not only from concrete experiences to classification, but to the learning of extremely complex abstract systems such as exist in mathematics and which more and more people, including children, are called upon to learn. The difficulties, in the way of such studies, are great, one of the main difficulties being that mathematicians on the whole are not interested in learning, and psychologists on the whole do not know enough mathematics to be able to formulate the problem in a way in which a possible solution might be sought. At the Sherbrooke Psychomathematics Center, for the past few years, we have been studying the abstraction process as it proceeds from the concrete to the final stage of wielding a mathematical formal system. We have rather few laboratory results of this aspect of our work as yet but we have enough classroom evidence that we can postulate certain regularities that seem to occur and certain pre-requisites that appear to have to be satisfied before certain stages of learning can successfully be undertaken. We have, so far, sub-divided the process of learning mathematics into 6 consecutive stages. These are approximately the following:

1. Initial inter-action.
2. Discovery of regularities in situations and consequent play with sets of rules or constraints.
3. The comparison of several games possessing the same structure, that is with the same sets of rules; this is the stage of the search for isomorphisms.
4. The representation of isomorphic situations in one, all-embracing, usually graphical form.
5. The study of the representation by the description of its properties.
this way not only one rule-structure or one game is being studied, but all
games with identical rule structures. This might be called the stage of symbo-
lizing, since in order to describe something, symbols will be necessary.

(6) Out of all the descriptions, certain parts are taken as initial descrip-
tions and rules derived whereby other parts of the description can be derived
from the initial description. The rules must be so constructed that any des-
scriptions derived do, in fact, describe properties of the representation of our
structure. The initial descriptions are known as axioms, the rules of the
game are usually called the rules of procedures, and the eventual descriptions
which we can reach from the initial axioms are known as theorems. The
method used to arrive at a theorem from the initial description, using the rules
of procedure, is known as a proof. This final stage could be called the stage
of formalization.

To sum up, the stages are the following:
(1) Inter-action.
(2) Rule construction and manipulation.
(3) Isomorphisms.
(4) Representation.
(5) Symbolization.
(6) Formalization.

In this paper, I shall try to give a geometrical example of how we can lead
a child from initial inter-action to the manipulation of a formal system.
I have chosen a very simple example, namely the equilateral triangle.

STAGE I

A triangle is the simplest two-dimensional polygon and the equilateral
triangle is the most symmetric version of that simple figure. So, it seems a
fitting introduction to many aspects of geometry. In effect, studying the
triangle is studying, in a sense, cycles of three. If we put a pin through the
centroid of a piece of cardboard cut off in the form of an equilateral triangle
and we trace its perimeter on the sheet on which we have placed it, we can
turn it around until the cardboard triangle again is seen to be inside the
perimeter we have drawn. When we have done this for the third time, the
cardboard triangle will assume its original position. So, a cycle of three is
one of the important properties that we shall have to encourage children
to abstract from the study of an equilateral triangle. There are many cycles
of three in real life. For instance, most people have three meals a day: break-
fast, lunch and supper. And they go to bed and the next day they again have
breakfast, lunch and supper, and so on. The properties of this cycle are very
similar to the properties of the cycle described by an equilateral triangle