Subharmonic Bifurcation in an $S$-$I$-$R$ Epidemic Model

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Abstract. An $S \rightarrow I \rightarrow R$ epidemic model with annual oscillation in the contact rate is analyzed for the existence of subharmonic solutions of period two years. We prove that a stable period two solution bifurcates from a period one solution as the amplitude of oscillation in the contact rate exceeds a threshold value. This makes rigorous earlier formal arguments of Z. Grossman, I. Gumowski, and K. Dietz [4].

Key words: Epidemic model — Subharmonic solution — Bifurcation

Before the introduction of vaccines, several infectious diseases such as measles, chickenpox and mumps were endemic and exhibited oscillatory levels of incidence in large population centers. Data from New York City for the period between 1948 and 1964 show annual outbreaks of chickenpox and mumps and a biennial cycle of measles incidence characterized by a large outbreak one year followed by a mild outbreak the next year [9].

Mathematical models of the spread of an infectious disease in large populations have been based on the law of mass action whereby the rate at which the subpopulation of susceptible individuals become infected is assumed to be proportional to the product of the proportion of susceptible individuals and the number of infectious persons in the population. The proportionality constant in the relation is referred to as the contact or transmission rate of the infection. Deterministic mathematical models attempting to describe the time evolution of the proportions of susceptible, infectious, and permanently immune individuals in the population typically yield damped oscillations about some mean value when the contact rate (as well as other parameters) is assumed to be constant in time [1, 3, 9, 11]. This is contrary to observation. London and Yorke [9] were the first researchers to attempt to infer the contact rate from existing data and their work shows that contact rates vary seasonally for chickenpox, mumps and measles. They also formulated a mathematical model in which they used their calculated contact rates to simulate yearly outbreaks of chickenpox and mumps and the biennial pattern of measles incidence. Their model consists of a system of delay differential equations which is difficult to analyze and consequently their work consists primarily of numerical simulations.

K. Dietz [3] introduced a mathematical model involving ordinary differential equations in which the contact rate is assumed to vary seasonally, which has...
proved more amenable to mathematical analysis. His numerical simulations indicate the possible existence of periodic solutions of periods one, two, four and six years for different parameter values. Focusing on the biennial pattern of measles, Gumowski et al. [4] formally (and nonrigorously) obtained periodic solutions of period two years for the Dietz model by perturbation methods (see also [5] where delays were introduced). The purpose of this paper is to give a rigorous proof of the existence of periodic solutions of period two years for the Dietz model.

We follow Dietz [3] in assuming that the population size is constant. The population consists of susceptible, infectious and recovered individuals. A latent period after becoming infected but before becoming infectious is ignored. New susceptibles are introduced at a constant rate, \( \mu \), by birth and all classes experience the same constant death rate, \( \mu \). The law of mass action, described above, is assumed to hold with a sinesoidal oscillation of the contact rate about a positive value, \( \beta \). A constant recovery rate, \( \gamma \), is assumed. The equations governing the proportions of susceptible, infectious and recovered individuals, denoted respectively by \( S \), \( I \) and \( R \), are:

\[
\begin{align*}
S' &= \mu - \mu S - \beta (1 + \delta \cos 2\pi t) IS, \\
I' &= \beta (1 + \delta \cos 2\pi t) IS - (\gamma + \mu) I, \\
R' &= \gamma I - \mu R, \\
S + I + R &= 1.
\end{align*}
\]

(0.1)

Notice that the first two equations suffice to describe the model. The time scale is in units of one year as is evident from the form of the contact rate, \( \beta (1 + \delta \cos 2\pi t) \).

Two epidemiologically relevant parameters are the infectious period \( P = 1/(\gamma + \mu) \) and the reproduction rate, \( Q = \beta P \). \( Q \) represents the number of secondary cases that an infected individual can produce in a population of susceptible individuals when \( \delta = 0 \). When \( \delta = 0 \), the equation (0.1) has a so called endemic steady state if \( Q > 1 \), given by

\[
\begin{align*}
S &= 1/Q, \\
I &= \mu(Q - 1)/\beta
\end{align*}
\]

(0.2)

which is globally asymptotically stable with respect to initial conditions in \( \{(S, I): S, I > 0, I + S < 1\} \) [8]. However, as pointed out by Grossman et al. [4], the stability is extremely weak for epidemiologically relevant values of the parameters in the sense that the eigenvalues of the linearized equation about (0.2) have small real parts compared to the imaginary part. In addition, it is plausible that for measles, the weakly damped oscillations about (0.2) have a nearly two-year quasi-period [3]. Thus, when \( \delta \neq 0 \) we have a resonance between the one year forcing and this natural period. The possibility of such a resonance was suggested by Dietz [3], exploited in Grossman et al. [4, 5], and will be the basis for our rigorous analysis. The existence of periodic solutions of arbitrary integral period for the Dietz model is discussed in [10] using entirely different methods.

In Sect. 1 one we scale variables in (0.1) and introduce several small parameters. Equation (1.2) in this section is one of the main results of this paper. In addition, the passive one year response of (0.1) to one year forcing is computed. Several changes of variable are made in order that the stability of the passive one year oscillation can