SEARCH STRATEGIES AS INDICATORS OF FUNCTIONAL THINKING

ABSTRACT. This paper presents an investigation into the development of functional thinking. Search strategies used for solving a problem reveal different stages towards the discovery of the monotonic property, which correspond to stages of proportional reasoning found by Piaget. The dependence between stage and age is confirmed by the results.

1. FUNCTIONAL THINKING

The concept of function is a "leading concept" in mathematics teaching. It is taught in secondary schools in a spiral approach in which students successively reach different levels of understanding (Vollrath, 1984). These levels are characterized by abilities, for example that the student is able to realize in a situation that a quantity \( y \) depends upon another quantity \( x \); or more specifically, the student knows that from an enlargement of \( x \) an enlargement of \( y \) will result. Such knowledge leads to assumptions when a new situation is investigated by the student. Once an observation differs from what is expected, learning can take place: a new assumption is made in order to explain the unexpected result.

To learn about functions and to be successful in using functions to solve problems requires a mental ability which can be characterized as follows:

(i) Dependences between variables can be stated, postulated, produced, and reproduced.
(ii) Assumptions about the dependence can be made, can be tested, and if necessary can be revised.

The mental activities described in (i) are fundamental for working on functions (Freudenthal, 1983). The activities in (ii) are typical for "mathematical thinking" (Burton, 1984). This ability can be called functional thinking, which has been a key concept in mathematics education since the Meran Conference in 1905 (Gutzmer, 1908). Many suggestions have been made by mathematics educators in Europe to promote functional thinking and there is some knowledge about the efficiency of these methods (e.g. Andelfinger, 1981; Hart, 1981).

Knowledge about the development of functional thinking can be gained by psychological studies. From Piaget's investigations it is known that the ability to discover the proportionality of a function develops in children (1958, 1968). He presented, for example, physical experiments and asked for the exact solutions of missing value problems which had to be found by

(C) 1986 by D. Reidel Publishing Company.
calculation, and identified stages leading to "proportional reasoning". (Several studies attempted to fill gaps in the Piagetian model e.g. Case, 1979; Ricco, 1982; Siegler, 1976. For research on proportional reasoning we refer the reader to Tourniaire and Pulos, 1985.) These stages can be interpreted as stages of functional thinking. They can be defined by abilities and by limitations, such as:

The child knows that from an enlargement of $x$ an enlargement of $y$ will result. But the child is not able to discover that a doubling of $x$ leads to a doubling of $y$.

These abilities and deficits become apparent in problem situations in which the student is asked to predict or to precalculate a result.

Refering to Freudenthal (1983) proportional reasoning has a number aspect (ratio) and a function aspect (proportionality). The function aspect underlies the "building-up" strategies (Hart, 1981) which are frequently observed during childhood and adolescence (Hart, 1981; Ricco, 1982). Changing strategies in an experiment can be understood as a result of changing assumptions about the dependence between the relevant quantities. When a missing value problem is presented in a physical experiment with an underlying quadratic function, many children assume proportionality, for example, and have difficulty overcoming this assumption (Suarez, 1977). To be successful in such problems it is therefore important to discover properties "beyond proportionality".

The influence of teaching methods on the ability to form assumptions about functions, used in physics instruction was investigated by Häussler (1981). Experienced students typically start with the assumption of proportionality, but they also have a repertoire of further properties of functions which they can use for testing assumptions. Research on functional thinking should be aimed at yielding more information about the development of such a repertoire. Inexperienced children tend to use "fallback" strategies when confronted with a new situation (Karplus et al., 1983). There seems to be a hierarchy between the properties in the repertoire, which depends on the problems presented.

Most of the problems in these investigations had to be solved exactly by calculations. But there are many problems in mathematics instruction where solutions have to be found by approximation; e.g. calculating the zeros of $y = x^2 - 2$ by iteration, or approximating the ratio of the circumference to the diameter of a circle. There is a growing interest in such algorithms because of the increasing importance of the computer. They can be understood as goal-seeking processes, and the methods used are called search strategies.