On the Super-KP Hierarchy

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Abstract. We obtain super-integrable systems from the super-KP hierarchy of Kac and van de Leur and give a method to construct solutions. In particular, we apply the method to get the super-version of the KP 1-soliton solution.


Key words: Kadomtsev–Petviashvili hierarchy, super-integrable systems, super-boson-fermion correspondence

1. Introduction

In the last decade, several super-symmetric extensions of the Kadomtsev–Petviashvili hierarchy have been introduced. Among these, the SKP hierarchy of Manin and Radul [10, 12, 15] the SKP hierarchy of Kac and van de Leur [7, 8] and the Jacobian hierarchy [11–13] have been given the most attention. Some relationships between them were studied by Rabin [14].

The SKP hierarchy of Kac and van de Leur is constructed using the super-boson-fermion correspondence [7, 8]. This super-boson-fermion correspondence is the natural super analogue of the classical boson-fermion correspondence, which is the main ingredient in the work of the Kyoto School on the KP hierarchy and soliton solutions [1–3, 5, 9]. Besides, the fermionic vertex operators appearing in this super KP are similar to those appearing in string theory [4].

In this Letter, we show how super-integrable systems can be obtained from the super-KP hierarchy of Kac and van de Leur (the equations obtained in [8] are not quite correct) and we give a method to construct solutions of this equations. The simplest nontrivial equations obtained are the following super-Hirota bilinear equations:

\[ (D_{-1}^4 + 3D_{-2}^2 + 8D_{-1}D_{-3} - 12D_{-1}D_0^2 D_2^2 + 12D_2^2 D_{-2}^2 - 8D_0^2 D_2^2 \cdot \tau_0(x, \theta) \cdot \tau_0(x, \theta) = 0, \]

(1.1)

\[ (D_{-1}^4 + 3D_{-2}^2 + 8D_{-1}D_{-3} - 12D_{-1}D_0^2 D_2^2 + 12D_2^2 D_{-2}^2 + 8D_0^2 D_{-3}^2 \cdot \tau_0(x, \theta) \cdot \tau_0(x, \theta) = 0 \]

(1.2)
where $x = (x_1, x_2, x_3)$ are even variables, $\theta = (\theta_{\pm 1}, \theta_{\pm 2}, \theta_{\pm 3})$ are odd variables, and

$$P(D, D_0)\tau \cdot \sigma = P \left( \frac{\partial}{\partial y}, \frac{\partial}{\partial \alpha} \right) \tau(x + y, \alpha) \sigma(x - y, \beta) \bigg|_{y = \alpha = \beta = 0}. $$

These equations reduce to the KP equation if the $\tau$-function $\tau_0(x, \theta)$ does not depend on the odd variables $\theta_{\pm i}, i = 1, 2, 3$. The simplest nontrivial solution for these equations, which reduces to the classical 1-soliton solution for the KP equation, is given by

$$\tau_0(x_1, x_2, x_3, \theta_{\pm 1}, \theta_{\pm 2}, \theta_{\pm 3}) = 1 + a e^{(x^2 - w_2)x_1 + (x^4 - w_4)x_2 + (x^6 - w_6)x_3} \times \left\{ x w \theta_{-1} \theta_1 + 2 x w^3 \theta_{-2} \theta_1 + 3 x w^5 \theta_{-3} \theta_1 + 2 x w \theta_{-1} \theta_2 + \right.$$

$$+ 4 x^3 w^3 + \theta_{-2} \theta_2 + 6 x^3 w^5 \theta_{-3} \theta_2 + 3 x^3 w \theta_{-1} \theta_3 + \right.$$

$$+ 6 x^5 w^3 \theta_{-2} \theta_3 + 9 x^5 w^5 \theta_{-3} \theta_3 - \frac{x w}{(x^2 - w^2)^2} \right\}.$$ 

(1.3)

2. Super-Boson-Fermion Correspondence and Super-KP Hierarchy

In this section, we recall results from [7] and [8] which lead to the super-extension of the KP hierarchy we are interested in.

Consider the Clifford super-algebra $\mathcal{C} = \mathcal{C}_0 \oplus \mathcal{C}_1$ on generators $\psi_i$ and $\psi_i^* (i \in \frac{1}{2} \mathbb{Z})$, which are called free super-fermions, with defining relations

$$\psi_i \psi_j + (-1)^{ij} \psi_j \psi_i = \psi_i^* \psi_j^* + (-1)^{4ij} \psi_j^* \psi_i^* = 0,$$

$$\psi_i \psi_j^* + (-1)^{4ij} \psi_j^* \psi_i = \delta_{ij}.$$ 

(2.1)

The $\mathbb{Z}_2$-gradation being given by

$$\psi_i, \psi_i^* \in \begin{cases} \mathcal{C}_0 & \text{if } i \in \mathbb{Z}, \\ \mathcal{C}_1 & \text{if } i \in \mathbb{Z} + \frac{1}{2}. \end{cases}$$

(2.2)

There exists an irreducible Cl-module (the spin module) $V$, which admits a nonzero vector $|0\rangle$, which is called the vacuum vector, satisfying

$$\psi_j |0\rangle = 0 \quad \text{for } j \leq 0, \quad \psi_j^* |0\rangle = 0 \quad \text{for } j > 0.$$ 

(2.3)

Elements

$$\psi_{i_1}^* \psi_{i_2}^* \ldots \psi_{i_r}^* \psi_{j_1} \psi_{j_2} \ldots \psi_{j_t} |0\rangle.$$ 

(2.4)