ON INFINITE DIRECT PRODUCTS OF LORENTZ TRANSFORMATIONS

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ABSTRACT. An infinite direct product $\otimes_{i=1}^{\infty} U_i(a)$ of continuous unitary representations of $SL(2, \mathbb{C})$ in Hilbert spaces $\mathcal{H}_i$ is continuous only on certain incomplete direct product subspaces of $\otimes_{i=1}^{\infty} \mathcal{H}_i$. If no representations of the complementary series occur, then each of these subspaces contains a product vector almost all factors of which are $SL(2, \mathbb{C})$-invariant.

Infinite direct products of Lorentz transformations arise if non-Fock representations of the Canonical Commutation Relations (CCR) for quantum fields are constructed by means of infinite direct products of Fock (sub)representations (see e.g. [1-4]). Although main applications of the present paper lie also in that context, infinite direct products of Lorentz transformations are treated here as independent objects.

Given a sequence of continuous unitary representations $U_i(a)$ of $SL(2, \mathbb{C})$ in Hilbert spaces $\mathcal{H}_i$ ($i = 1, 2, ...$), there is a unitary representation

$$U(a) = \bigotimes_{i=1}^{\infty} U_i(a)$$

in von Neumann's infinite direct product space $\otimes_{i=1}^{\infty} \mathcal{H}_i$ (see [5] for $\otimes_{i=1}^{\infty} \mathcal{H}_i$, [6] for direct products of operators in that space). However, $U(a)$ acts continuously only on certain subspaces of $\otimes_{i=1}^{\infty} \mathcal{H}_i$ the characterization of which is the purpose of this paper.

According to [5], $\otimes_{i=1}^{\infty} \mathcal{H}_i$ is spanned by finite linear combinations of product vectors $\otimes_{i=1}^{\infty} \Phi_i$ satisfying the condition

$$\sum_{i=1}^{\infty} |1 - \|\Phi_i\|^2| < \infty.$$  

The inner product on $\otimes_{i=1}^{\infty} \mathcal{H}_i$ is defined by linear continuation of

$$\bigotimes_{i=1}^{\infty} \bigotimes_{i=1}^{\infty} \langle \Phi_i, \Psi_i \rangle = \prod_{i=1}^{\infty} (\Phi_i, \Psi_i)$$

(cf. [5] for quasi-convergence of the infinite product), whereas $\otimes_{i=1}^{\infty} U_i(a)$ is defined by linear continuation of

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Besides the 'complete' direct product $\bigotimes_{i=1}^{\infty} \mathcal{H}_i$ von Neumann also introduced 'incomplete' direct product spaces (IDPS) $\bigotimes_{i=1}^{\infty} (\mathcal{H}_i, \Phi_i)$ which are separable subspaces of $\bigotimes_{i=1}^{\infty} \mathcal{H}_i$. An IDP space with reference vector $\bigotimes_{i=1}^{\infty} \Phi_i$ contains only linear combinations of product vectors $\bigotimes_{i=1}^{\infty} \Psi_i$ satisfying

$$\sum_{i=1}^{\infty} \| 1 - (\Phi_i, \Psi_i) \| < \infty. \quad (5)$$

As shown in the Appendix, (weak) continuity of $U(a)$ on some $\Psi \in \bigotimes_{i=1}^{\infty} \mathcal{H}_i$ implies strong continuity of $U(a)$ on any product vector $\bigotimes_{i=1}^{\infty} \Phi_i$ 'contained' in $\Phi$:

**PROPOSITION 1.** If $(\Psi, U(a)\Psi)$ with $\Psi \in \bigotimes_{i=1}^{\infty} \mathcal{H}_i$ is continuous at some $a_0 \in SL(2, C)$, and $(\Psi, \bigotimes_{i=1}^{\infty} \Phi_i) \neq 0$, then $\bigotimes_{i=1}^{\infty} (\mathcal{H}_i, \Phi_i)$ is invariant under $U(a)$ for all $a \in SL(2, C)$, and $U(a)$ is a strongly continuous representation of $SL(2, C)$ in $\bigotimes_{i=1}^{\infty} (\mathcal{H}_i, \Phi_i)$.

In order to find subspaces of $\bigotimes_{i=1}^{\infty} \mathcal{H}_i$ carrying a continuous unitary representation of $SL(2, C)$ it is therefore sufficient to consider the action of $U(a)$ on product vectors.

Because of (1) and (5), $SL(2, C)$-invariance of an IDPS $\bigotimes_{i=1}^{\infty} (\mathcal{H}_i, \Phi_i)$ implies

$$\sum_{i=1}^{\infty} \| 1 - (\Phi_i, U_i(a)\Phi_i) \| < \infty \quad \text{for all} \quad a \in SL(2, C). \quad (6)$$

In drawing conclusions from (6) the classification of irreducible unitary representations of $SL(2, C)$ into the trivial representation, the principal and the complementary series (see e.g. [7]) turns out to be useful.

For each $i$ let $\Phi_i$ be decomposed according to

$$\Phi_i = \Phi_i^0 \oplus \Phi_i^p \oplus \Phi_i^c, \quad (7)$$

where $\Phi_i^0$ is the $SL(2, C)$-invariant part of $\Phi_i$, and $\Phi_i^p$ and $\Phi_i^c$ are direct integrals over representations of the principal and complementary series, respectively. The $0$, $p$ and $c$ parts of $\Phi_i$ are mutually orthogonal, and transform separately under $U_i(a)$.

**PROPOSITION 2.** Equation (6) implies $\sum_{i=1}^{\infty} \| \Phi_i^p \|^2 < \infty. \quad (8)$

(Proof see below.)

This means that, for $i \to \infty$, $\Phi_i$ 'essentially' consists of an $SL(2, C)$-invariant part and a part which transforms according to representations of the complementary series only.

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