NOETHER'S THEOREM AND STEUDEL'S CONSERVED CURRENTS FOR THE SINE-GORDON EQUATION

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ABSTRACT. A version of Noether's theorem appropriate for the extended Hamilton–Cartan formalism for regular first-order Lagrangians is proposed. Steudel's derivation of an infinite collection of conserved currents for the sine-Gordon equation is presented in this context and it is demonstrated that, as a consequence of the commutativity of the sine-Gordon Backlund transformations, the conserved charges corresponding to these currents are in involution with respect to the natural Poisson bracket provided by the formalism. Thus one obtains the formal 'complete' integrability of the sine-Gordon equation as a consequence of the properties of the Backlund transformation.

1. INTRODUCTION

In [6] and [7] the Hamilton–Cartan formalism for regular first-order Lagrangians was extended to deal with higher-order conserved currents, that is, conserved currents which involve derivatives of order higher than those which appear in the Lagrangian. It was demonstrated that to each such conserved current is associated a vector field which preserves the classical action. Thus the vector field may be regarded as a Noether symmetry and the correspondence between conserved currents and symmetries may be viewed as analogous to the converse of the usual version of Noether's theorem [9, 10]. In Section 2 of the present paper, this point of view is adopted and a formulation of Noether's theorem in the context of the extended Hamilton–Cartan formalism is proposed.

It is well known that one may use the Backlund transformation for the sine-Gordon equation to derive an infinite collection of higher-order conserved currents [3, 5, 9, 11] and in [9] Steudel showed that an extended Backlund transformation could be used to obtain such a collection as Noether currents. This result is presented in the context of the Hamilton–Cartan formalism in Section 3. It is demonstrated that the conserved charges determined by these currents are all in involution with respect to the Poisson bracket determined by the formalism. Thus the sine-Gordon equation possesses infinitely many conserved charges in involution and is formally completely integrable. Furthermore, it is shown that this is a simple consequence of the commutativity of the extended Backlund transformations taken together with the Lie algebra isomorphism [6, 7], between the algebra of conserved charges and the algebra of the corresponding vector fields.
2. NOETHER'S THEOREM AND ITS CONVERSE IN THE EXTENDED HAMILTON-CARTAN FORMALISM

In this section notational conventions are set and a theorem establishing the correspondence between higher order conserved currents and infinitesimal symmetries is stated. A version of Noether's theorem appropriate in the context of the extended Hamilton-Cartan formalism is presented and a proof is given.

Notation

In what follows, all geometric objects are smooth, i.e. $C^\infty$. If $M$ is a manifold then $\mathfrak{X}(M)$ and $\Lambda^p\Gamma^*(M)$ denote the vector fields on $M$ and the differential $p$-forms on $M$, respectively. If $X$ is a vector field and $\xi$ is a differential form then $X(\xi)$ denotes the Lie derivative of $\xi$ with respect to $X$, and $X \lrcorner \xi$ denotes the interior product of $\xi$ with $X$.

Let $M$ be an oriented manifold of dimension $m$, with local coordinates $x^a$ and volume $m$-form $\omega$ given in these coordinates by $\omega = dx^1 \wedge \ldots \wedge dx^m$. Let $N$ be an $n$-dimensional manifold with local coordinates $z^A$ and let $(E, \pi, M)$ be a fibre bundle with fibre $N$. The $k$ jet bundle of local sections of $(E, \pi, M)$ is denoted $J^k(E)$ and $J(E)$ denotes the infinite jet bundle \[4\]. The canonical projections from $J^k(E)$ to $M$ and from $J(E)$ to $M$ are denoted $\pi^k_M$ and $\pi_M$, and $\pi_k$ is the canonical projection from $J(E)$ to $J^k(E)$. The $k$th order and infinite-order contact modules are denoted $\Omega^k$ and $\Omega^\infty$.

The standard bases for $\Omega^k$ and $\Omega^\infty$ are

$$\{\partial^A, \partial^A_{a^1...a^j}, \partial^A_{a^1...a^j} \}$$

where, in standard local coordinates,

$$\partial^A = \frac{\partial}{\partial z^A} - z^A \frac{\partial}{\partial x^b} \quad \text{and} \quad \partial^A_{a^1...a^j} = \frac{\partial}{\partial z^{A_{a^1...a^j}}} - z^{A_{a^1...a^j}} \frac{\partial}{\partial x^b} \text{.}$$

The modules of contact $m$-forms $\Omega^k_{(m)}$ and $\Omega^\infty_{(m)}$ are the modules generated by $\Omega^k \wedge \{\omega_a\}$ and $\Omega^\infty \wedge \{\omega_a\}$ where $\omega_a$ is the $(m - 1)$-form defined by $\omega_a = (\partial/\partial x^b) \lrcorner \omega$.

If $s$ is a local section of $(E, \pi, M)$ then $J^k s$ and $J s$ are the jet extensions of $s$. Clearly $J^k s \ast \Omega^k_{(m)} = 0$ and $J s \ast \Omega^\infty_{(m)} = 0$.

The Lie algebra $\mathfrak{D}$ of total derivative vector fields on $J(E)$ is defined by

$$\mathfrak{D} = \{ X \in \mathfrak{X}(J(E)) \mid X \lrcorner \Omega = 0 \}$$

and has a basis $\{ D_a \}$ given in standard local coordinates by

$$D_a = \frac{\partial}{\partial x^a} + z^A_{a} \frac{\partial}{\partial z^A} + \ldots + z^A_{a^1...a^j} \frac{\partial}{\partial z^{A_{a^1...a^j}}} + \ldots \quad (2.1)$$

Let $\mathfrak{G}_V$ denote the Lie algebra of vector fields on $J(E)$ defined by

$$\mathfrak{G}_V = \{ X \in \mathfrak{X}(J(E)) \mid X(\Omega) \subseteq \Omega \quad \text{and} \quad \pi_M \ast X = 0 \}.$$ 

Clearly the vector fields in $\mathfrak{G}_V$ preserve the contact $m$-forms on $J(E)$ as well as preserving $\Omega$, that is, if $X \in \mathfrak{G}_V$ then

$$X(\Omega_{(m)}) \subseteq \Omega_{(m)} \quad (2.2)$$

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