On the Mean Width of Random Polytopes

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Summary. On the boundary of a $d$-dimensional convex body a probability distribution with a positive, continuous density function $g$ is given. The convex hull of $n$ points chosen independently according to $g$ is a random polytope. The asymptotic behaviour ($n \to \infty$) of the expected value of the mean width of the random polytope is determined.

Let $K$ be a $d$-dimensional convex body in $\mathbb{R}^d$, and let $W(K)$ denote the mean width of $K$. A random polytope is the convex hull $H_n$ of $n$ independent and identically distributed random points chosen from the interior $\text{int} \ K$ of $K$ or from the boundary $\partial K$ of $K$. The mean width of $H_n$ is a random variable with expected value $EW_n$.

The asymptotic behaviour, as $n \to \infty$, of $W(K) - EW_n$ is known, if the $n$ random points are chosen according to one of the following probability distributions:

(i) uniform distribution on $\text{int} \ K$ for smooth $K$ (Schneider and Wieacker [10]);
(ii) distribution on $\text{int} \ K$ for smooth $K$ with a density function, which is positive on $\partial K$ and Lipschitz-continuous near $\partial K$ (Ziezold [11]);
(iii) uniform distribution on $\partial K$ for smooth $K$ (Buchta et al. [3]);
(iv) uniform distribution on $\text{int} \ K$ for a convex polytope $K$ (Schneider [8]).

If the $n$ random points are chosen uniformly from $\text{int} \ K$ for arbitrary $K$, Schneider [8] gives lower and upper bounds for $W(K) - EW_n$, which show that in some sense the approximation of arbitrary $K$ by random polytopes is not worse than for polytopal $K$ and not better than for smooth $K$ (cases (iv) and (i)). If $K$ is the $d$-dimensional unit ball, in cases (i) and (iii) $EW_n$ is explicitly determined for fixed $n$ by Buchta et al. [2], [3].

In the present note, (iii) is generalized to a wider class of distributions. This result has independently been obtained by Prof. R. Schneider (unpublished). Buchta [1] and Schneider [9] give surveys on many results concerning random polytopes.
Theorem. Let $K$ be a convex body in $\mathbb{R}^d$ with boundary $\partial K$ of class $C^3$ and positive Gaussian curvature $\kappa$. Denote by $H_n$ the convex hull of $n$ random points chosen independently according to a probability distribution on $\partial K$, which has a density function $g$ with respect to the surface area measure $\sigma$ on $\partial K$. Let $EW_n(g)$ denote the expected value of the mean width of $H_n$, then

$$\lim_{n \to \infty} (W(K) - EW_n(g)) \cdot n^{2/(d-1)} = \frac{2}{(d-1) \omega_d} \Gamma \left( \frac{2}{d-1} \right) \pi^{d-1} \int_{\partial K} \kappa^{d-1} g^{-2/(d-1)} d\sigma,$$

if $g$ is positive and continuous on $\partial K$. Here $\pi_d$ is the volume and $\omega_d$ is the surface area of the $d$-dimensional unit ball.

Proof. In the first part of the proof we follow Schneider and Wieacker [10]. We suppose that the origin of the coordinate system is an interior point of $K$. We denote by $S_{d-1}$ the $(d-1)$-dimensional unit sphere and by $\omega$ the surface measure on $S_{d-1}$. A hyperplane $H(u, p)$ in $\mathbb{R}^d$ is given by a unit normal vector $u \in S_{d-1}$ and the oriented distance $p$ to the origin. The mean width of the convex hull $H_n = H_n(x_1, \ldots, x_n)$ of the points $x_1, \ldots, x_n \in \partial K$ is given by

$$W_n = \int_{S_{d-1}} \int_{-\infty}^{\infty} I(H_n, u, p) \, dp \, d\omega(u)$$

where

$$I(H_n, u, p) = \begin{cases} 1 & \text{if } H_n \cap H(u, p) \neq \emptyset \\ 0 & \text{else} \end{cases}$$

So

$$EW_n(g) = \int_{S_{d-1}} \int_{-\infty}^{\infty} M(u, p) \, dp \, d\omega(u),$$

where

$$M(u, p) = \int_{x_1 \in \partial K} \cdots \int_{x_n \in \partial K} I(H_n, u, p) g(x_1) \, d\sigma(x_1) \cdots g(x_n) \, d\sigma(x_n)$$

is the measure of the set $\{(x_1, \ldots, x_n) \in (\partial K)^n : H_n \cap H(u, p) \neq \emptyset\}$ for a fixed hyperplane $H(u, p)$.

Let $h(u)$ denote the support function of $K$. If $p \not\in [-h(-u), h(u)]$, we have $M(u, p) = 0$. Otherwise $H(u, p)$ cuts $\partial K$ into two pieces having probability measures $S_g(u, p) \leq \frac{1}{2}$ and $1 - S_g(u, p)$ respectively. In this case

$$M(u, p) = 1 - S_g(u, p)^n - (1 - S_g(u, p))^n.$$

We get

$$EW_n(g) = W(K) - 2 \int_{S_{d-1}} \int_{0}^{h(u)} (1 - S_g(u, p))^n \, dp \, d\omega(u) + O((\frac{1}{n})^n) \quad \text{as } n \to \infty.$$