The reference coordinates and distortion measures for quadrilateral hybrid stress membrane element

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Abstract The skew Cartesian coordinate system determined by the Jacobian of the isoparametric transformation evaluated at the origin can be shown to be a geodesic coordinate system at the origin. By using a theory in differential geometry, inverse relations of the isoparametric coordinate transformation can be derived and expressed in terms of these geodesic coordinates. In the formulation of hybrid stress finite elements, it is suggested as a new strategy for assumed stresses that such coordinates be used as the reference coordinates. The theory described is exemplified by its applications to the 4-node hybrid stress membrane elements. A set of new distortion-measuring parameters for the quadrilateral element are also proposed based on such theory.

1 Introduction
In the developments of hybrid stress finite elements, different coordinate systems have been used to express the assumed stresses. In the early days, local Cartesian system, global Cartesian system, or local Cartesian system rotated to better approximate the isoparametric coordinate system have been adopted in order for stresses to satisfy the equilibrium conditions (Pian 1964; Tong and Pian 1969; Cook 1974; Spilker et al. 1981). After the Hellinger-Reissner principle be suggested as the formulation basis (Pian and Chen 1982), the isoparametric coordinates become currently employed. Punch and Atluri (1984, 1984a) have proposed, in their a posteriori centroidal stress concept, that the stresses be expressed in terms of the isoparametric coordinate variables but the tensor basis of stress be determined by the centroidal base vectors of the isoparametric coordinate system. Recently, Sze and Chow (1991) introduced further a skew rectilinear coordinate system, which is obtained by projecting the local Cartesian system onto the axes determined by the centroidal base vectors of the isoparametric coordinate system. In a development similar to the hybrid stress element, Robinson (1975, 1985) has also employed a skew Cartesian coordinate system in expressing the oblique, instead of the Cartesian, stresses in the construction of his stress-based membrane and plate bending elements.

Assumed stresses in the modern hybrid stress elements initially need not be chosen to satisfy the equilibrium conditions pointwise. Based on the Hellinger-Reissner principle, additional incompatible displacements may be used as the Lagrange multipliers to constrain them in a variational sense. In a recent study of the 4-node membrane element, however, Yuan et al. (1993) have found that by using a skew Cartesian coordinate system determined by the Jacobian of the mapping evaluated at the origin, the assumed stresses can be made to satisfy the equilibrium conditions pointwise. Consequently, either the Hellinger-Reissner or the complementary energy functional can be used as the basis of element formulation. The skew Cartesian coordinate system used in Yuan et al. (1993) is linearly related to the local Cartesian coordinate system through the Jacobian of the mapping evaluated at the origin, denoted by $J_0$. In the literature, this particular Jacobian $J_0$ has appeared in the studies of many incompatible and hybrid/mixed elements (for example, Taylor et al. 1976; Pian and Sumihara 1984; Punch and Atluri 1984; Lu et al. 1989; Simo and Rifai 1990; Sze et al. 1990). However, the mathematical meaning of using $J_0$ in the formulations of such elements has not been clearly explored. This paper shows, by using the theory

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of geodesics in differential geometry (Eisenhart 1949; Veblen 1962), that the significance of \( J_0 \) lies in that it defines a geodesic coordinate system associated with the isoparametric mapping at the origin.

For the 4-node plane element, such a geodesic coordinate system is the skew Cartesian coordinate system used in Yuan et al. (1993).

By using the theory of geodesics in differential geometry, inverse relations of the mapping for the 4-node element can also be derived in the form of power series in these geodesic coordinates. The coefficients in such series, and those in the polynomial of the determinant of the Jacobian, can all be determined by the values at the origin of the two non-vanishing Christoffel symbols of the isoparametric coordinate system. These two values, together with the aspect ratio and skew angle of the parallelogram determined by the axes of the skew Cartesian coordinate system, are proposed to be the distortion measures of the element. The use of them in describing the effect of geometric distortion on the strain field for the 4-node element will also be discussed.

The theory described in this paper is completely general. The geodesic coordinates determined by \( J_0 \) can thus be used as the reference coordinates for assumed stresses in the formulation of any 2-D, 3-D or axisymmetric hybrid stress elements.

2

Theory of geodesics

The geodesics is the curve of minimum length joining any two given points in a curved Riemannian space. In the special case of flat Euclidean space, it is simply a straight line. Passing through any point \( p \) in a general coordinate system \( \xi^i, i = 1 \rightarrow n \), there are infinitely many geodesic curves which are integral curves of the following system of differential equations (Eisenhart 1949).

\[
\frac{d^2 \xi^i}{ds^2} + \Gamma^i_{jk} \frac{d \xi^j}{ds} \frac{d \xi^k}{ds} = 0, \quad i = 1 \rightarrow n \tag{1}
\]

where \( s \) is the arc length parameter and \( \Gamma^i_{jk} = \Gamma^i_{kj} \) denotes the Christoffel symbols of the \( \xi^i \) coordinates.

Any integral curve of Eq. (1) passing through \( q \) can be uniquely determined by a direction at \( q \). Specifying the direction by

\[
t^i = \left( \frac{d \xi^i}{ds} \right)_q, \tag{2}
\]

one can expand the equations of the integral curve into Taylor series about \( q \).

\[
\xi^i = (\xi^i)_q + t^i s + \frac{1}{2} \left( \frac{d^2 \xi^i}{ds^2} \right)_q s^2 + \frac{1}{3!} \left( \frac{d^3 \xi^i}{ds^3} \right)_q s^3 + \cdots. \tag{3}
\]

To determine the coefficients of the higher powers in \( s \), one may first differentiate Eq. (1) successively to obtain the following sequence of equations, which must be satisfied by any solution curve.

\[
\frac{d^3 \xi^i}{ds^3} + \Gamma^i_{jk} \frac{d \xi^j}{ds} \frac{d \xi^k}{ds} + \frac{d \xi^i}{ds} = 0
\]

\[
\frac{d^4 \xi^i}{ds^4} + \Gamma^i_{jkl} \frac{d \xi^j}{ds} \frac{d \xi^k}{ds} + \frac{d \xi^i}{ds} + \frac{d \xi^i}{ds} = 0
\]

\[
\vdots
\]

where the functions

\[
\Gamma^i_{jk} = \frac{1}{3} P \left( \frac{\partial \Gamma^i_{jk}}{\partial \xi^l} - \Gamma^i_{sk} \Gamma^s_{jl} - \Gamma^i_{sj} \Gamma^s_{kl} \right) = \frac{1}{3} P \left( \frac{\partial \Gamma^i_{jk}}{\partial \xi^l} - 2 \Gamma^i_{sj} \Gamma^s_{kl} \right)
\]

and, in general,

\[
\Gamma^i_{jkl \cdots mn} = \frac{1}{N} P \left( \frac{\partial \Gamma^i_{jkl \cdots mn}}{\partial \xi^p} - \Gamma^i_{stk \cdots mn} \Gamma^s_{jl} - \cdots - \Gamma^i_{jl \cdots m} \Gamma^m_{kn} \cdots \right) \tag{5}
\]

in which \( P \rangle \rangle \) denotes the sum of terms obtainable from the ones inside the bracket by permuting the free subscripts cyclically and \( N \) denotes the number of subscripts. The functions \( \Gamma^i_{jkl \cdots mn} \) so defined