Boundary integral equation method for elastic beams in frictionless contact problems

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Abstract. A boundary integral equation algorithm for the contact analysis of elastic beams is presented in this paper. The analysis of this sort is complicated by the unknown and moving boundary points. The new algorithm incorporates the interface compatibility equation, derived from the principle of minimum potential energy, into the general boundary integral equation so that the locations of boundary points may be correctly identified. The incremental and iterative solution procedure is presented. Accuracy and efficiency of the new algorithm are demonstrated using examples whose classical solutions exist.

1 Introduction

The analysis of contact problems of solid continua is a common concern in the stamping, steel rolling and many other industries. Much of the difficulty of such analyses lies in the fact that the boundary conditions of the solid bodies considered are valid on the boundaries whose locations remain unknown and move in uncertain manners. The problems of this sort are frequently called the moving or free boundary problem. Various techniques have been developed to tackle the problems using finite elements (Campos et al. 1982; Pian and Kubomura 1980; Haber and Hariandja 1985; Gu 1986, 1988; Westbrook 1982) and boundary integral equation method (Andersson 1981; Karami and Fenner 1987) (also known as boundary element method). Most of the techniques implemented to tackle the contact problems normally require greatly refined meshes in order to predict the precise extent of contact region. Besides, the geometric compatibilities of node-to-node and edge-to-edge in contact region are in general violated, causing inaccurate estimates of deformation and contact stresses. Haber and Hariandja (1985) have incorporated the Eulerian displacements in the candidate contact nodes so that they may move along the undeformed contact surfaces and that the geometric compatibilities are remained. The two-dimensional contact problems, involving large deformation, frictional slip and flat contact surfaces, were studied in (Haber and Hariandja 1985). Westbrook (1982) suggests a boundary adjustment routine so that boundary nodes of a contact region can be relocated at the more accurate positions after the preliminary solutions are obtained. He investigates the frictionless contact between an elastic beam and a flat rigid barrier.

Recently Gu (1986, 1988) introduced the interface compatibility equation mechanism which precisely locates the marginal (boundary) nodes of the contact region and controls the movements of the marginal nodes as the external loads alter. All the finite elements and nodes in the elastic bodies are traveling relatively to the marginal nodes. The primary aim in this study is to incorporate such a mechanism in the boundary integral equation method with which we may obtain the accurate solutions to the extent of contact regions and the deflections of the elastic beams under various contact conditions. Although only elastic beams are investigated at this stage, it is believed that this new algorithm may be extended to other complicated solid continua such as plates.

This paper starts with the variational principle of minimum potential energy. Upon invoking the principle, the interface compatibility equations at marginal nodes are obtained. The description of the numerical implementation of the interface equation then follows. Several numerical
examples are performed using the proposed algorithm, whose results are then compared to those obtained from the classical beam theory and from the moving finite element algorithm (Gu 1986, 1988) with regard to computational effectiveness and accuracy.

2 Mathematical formulation

2.1 Interface Compatibility Equation

Figure 1 depicts the problem considered, in which $Q(x)$ is the lateral load, $f(x)$ describes the geometry of the rigid barrier, $a_1$ and $a_2$ are the unknown $x$-coordinates of the marginal points of the contact region of the deflected beam, and $L$ is the total length of the beam. Although shown as a distributed force, $Q(x)$ may include concentrated forces and moments via singularity functions of various orders. The rigid barrier may be either stationary or moving vertically in a prescribed rate. Only one rigid barrier is considered in the following treatment, although it may be implemented to include multiple contact bodies as in the example shown later.

Within the framework of Euler–Bernoulli classical beam theory, the functional of the total potential energy follows

$$\Pi = \int_0^a \frac{d^2w}{dx^2} \left( \frac{EI}{2} \frac{d^2w}{dx^2} \right) dx + \int_{a_1}^{a_2} \frac{d^2\tilde{w}}{dx^2} \left( \frac{EI}{2} \frac{d^2\tilde{w}}{dx^2} \right) dx + \int_0^L \frac{d^2w}{dx^2} \left( \frac{EI}{2} \frac{d^2w}{dx^2} \right) dx - \int_0^L Qw dx$$

(1)

where $E$ is the modulus of elasticity, $I$ is the moment of inertia, and $\tilde{w}$ and $w$ designate, respectively, the deflections of the beam in contact and non-contact regions. It is assumed here that the frictional force and tangential sliding between the beam and the rigid barrier within the contact region do not exist.

It needs to point out that the deflection $\tilde{w}$ inside the contact region may be determined from the given function $f(x)$, and the prescribed ascending rate of the barrier, if any. As a consequence, upon invoking the principle of minimum potential energy on the functional $\Pi(w, a_1, a_2)$, and using the variational technique (Shames and Dym 1985), one may derive the well-known fourth order differential equation governing the deflection of beam in non-contact regions together with some boundary conditions and the interface compatibility equation at the unknown marginal nodes of the non-contact region. In essence, the interface compatibility equation,

$$\frac{d^2w}{dx^2} = \frac{d^2\tilde{w}}{dx^2} \quad \text{at} \quad x = a_i \quad (i = 1, 2)$$

(2)

obtained by minimizing $\Pi(w, a_1, a_2)$ with respect to $a_i$, states the continuity of the curvature of the beam at the interface of contact and non-contact regions, which is to be utilized later to calculate the movement of the marginal node at an interface.

2.2 Direct Boundary Integral Method

It is a common practice in boundary integral method that one may obtain the deflection $w(\xi)$ at a field point $\xi$ of the beam by integrating by parts the fourth order governing equation.

![Fig. 1. Schematic representation of problem considered](image-url)