Enhanced lower-order element formulations for large strains

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Abstract The paper describes a range of lower-order element formulations that can be applied to both elastic and elasto-plastic large-strain elements. For plane-strain analysis, this process involves four-noded quadrilaterals while the enhancements involve incompatible modes or enhanced strains. One particular new formulation can be considered as either a "co-rotational approach" or a modified form of "Biot stress procedure".

1 Introduction

Since the pioneering finite element work of Hibbitt, Marcal and Rice (1970) and of Mc-Meeking and Rice (1975), there have been very many publications in relation to plasticity with large strains. Early work started with a rate form and involved an additive decomposition of the elastic and plastic strain rates (Atluri (1984)). More recently, one of the main trends has involved the increasing use of a multiplicative decomposition of the deformation gradient (Cuitiño and Ortiz (1992), Eterovic and Bathe (1990), Perić, Owen and Honnor (1992)). Key contributions have been made by Simo and co-workers (Simo and Ortiz (1985), Simo (1992)) and the links with rate formulations have been discussed by Needleman (1985) and Atluri (1986) amongst others. In addition, recent developments tend to treat the elastic deformation using a hyperelastic relationship (as in previous references and in Bathe, Slavković and Kojić (1986), Rolph III and Bathe (1984)). The main objective has not been to consider large elastic strains (although this can be done) but rather to avoid some issues regarding integration of the rate equations and the subsequent linearisation of the resulting algorithms.

Since a hyperelastic formulation is at the core of these elasto-plastic formulations, we will start the paper by considering such a hyperelastic formulation in its own right (Simo and Taylor (1991)), the key objective is to produce an effective formulation for low order elements which has the advantage of being easily implemented with existing contact algorithms. Unfortunately, in their pure forms such lower-order elements give a poor numerical performance and, in particular, lock when the response is almost volume-preserving (as for hyperelastic and elasto-plastic problems).

In the current work we will be mainly concerned with plane-strain (although the concepts are easily extended to a full 3-D analysis) and, in particular, the four noded quadrilateral. In a linear context, one of the most effective ways to enhance the performance of the four-noded element is to use the "incompatible mode formulation" of Taylor, Beresford and Wilson (1976). Simo and Rifai (1990) showed that a more general procedure could be devised with the aid of "mixed assumed strain procedures". Work on the application of these techniques to small-strain plasticity has been given by Andelfinger, Ramm and Roehl (1992) and to large-strain analysis by Simo and Armero (1992) and Simo, Armero and Taylor (1993).

While the enhanced strain concept is more general, in relation to the first four incompatible modes that are often used (Taylor, Beresford and Wilson (1976)), the two formulations are effectively identical. Therefore, in the following we will refer to "incompatible modes", although equivalent results could be obtained with "enhanced strains".

The extension to non-linearity by Simo and Armero (1992) and Simo, Armero and Taylor (1993) involved an approach on the current configuration using Kirchhoff (or Cauchy) stresses with an enhancement being made to the deformation gradient. An alternative approach has been described by the first two of the current authors which involves a form of "co-rotational technique" (Crisfield and Moita (1995), Crisfield and Moita (1994), Moita (1994)) which has some concepts in common with an earlier large-strain co-rotational technique of Jletteur and Cescotto (1991). In a continuum context the co-rotational technique has very close links with a Biot-stress formulation and we will establish these links in this paper. Particularly, once the co-rotational technique is extended to large-strain plasticity, there are some advantages in considering the co-rotational
technique in this continuum rather than in the usual "structural manner".

In the paper, we will develop the co-rotational formulation and, in some circumstances, will compare the solutions with those obtained via an implementation of the enhanced deformation gradient approach of Simo and Armero (1992) and Simo, Armero and Taylor (1993). In some respects, advantages will be claimed for our co-rotational approach. However, in a work dedicated to the memory of Juan Simo, some key provisions should be pointed out. In particular, we believe that in the recent past any research worker in computational mechanics aspiring to good work, had to attempt (almost certainly vainly) to compete with Juan Simo. In part, the attempt was vain because there was a continually moving target. Sadly, this is no longer so. Nonetheless, it almost goes without saying that the work we will describe would be impossible without the foundations laid by Juan Simo.

2 The co-rotational approach

The co-rotational approach was initially introduced by Wempner (1969) and Belytschko and Hsieh (1973) and has much in common with the "natural approach" of Argyris, Balmer, Doltisnis, Dunne, Haase, Kleiber, Malejannakis, Mlejnek, Müller and Scharpf (1979). The method is normally applied to beams and shells, so that the definition of a local base-frame is fairly straightforward. This is no longer true, once the method is applied to continuum. A wrong choice of local base frame will mean that the element will not pass the "large-strain patch test" (Jetteur and Cescotto (1991) and Simo and Armero (1992)). Consequently, we have followed Jetteur and Cescotto in choosing the local axes such that the local spin at the centroid of the element in the current configuration should be zero, i.e.

$$\Omega = \frac{\partial u_i}{\partial X_i} - \frac{\partial \nu_i}{\partial X_i} = 0$$

(1)

With respect to Fig. 1, the rotating local frame can be defined from the co-rotating base vectors $e_1$ and $e_2$ expressed in terms of the angle $\theta$ so that

$$e_1 = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}, \quad e_2 = \begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix}$$

(2)

Using these vectors we can apply two alternative methods to define the current configuration of node $i$ so that

$$x' - x_i = (x_i + p_i) - (X_i + p'_i) = [e_i, e_j] (X'_i + p'_i)$$

$$= E(X'_i + p'_i)$$

(3)

where, if there is no subscript $i$, the quantity is global and $x$ defines the current co-ordinates and $X$ the initial co-ordinates. The vector $p'$ contains the global total displacements at node $i$ (i.e. $u'$ and $v'$) while $x'$ has the current co-ordinates of node $i$, $x'$ and $y'$. It should be noted that $X'_i = X_i - X_i$ so that with respect to a new origin at node 1, the initial co-ordinates in the local and global systems are the same (see Fig. 1). If there is a rigid-body motion about node 1 so that, for simplicity, we can set $x' = X_i = p_i = 0$, we obtain $x' = E X'$ so that $E'$ then takes the form of a rotation matrix.

Equation (1) for the local spin can be expressed in terms of the shape functions and the local displacements, $p_i$, as

$$\Omega_i = a_i^T (X) p_i$$

(4)

where

$$a_i = \begin{bmatrix} (J_0^{-1}(2,1)h_i^2 + J_0^{-1}(2,2)h_i^4) \\
-(J_0^{-1}(1,1)h_i^2 + J_0^{-1}(1,2)h_i^4) \\
(J_0^{-1}(2,1)h_i^2 + J_0^{-1}(2,2)h_i^4) \\
-(J_0^{-1}(1,1)h_i^2 + J_0^{-1}(1,2)h_i^4) \\
(J_0^{-1}(2,1)h_i^2 + J_0^{-1}(2,2)h_i^4) \\
-(J_0^{-1}(1,1)h_i^2 + J_0^{-1}(1,2)h_i^4) \end{bmatrix}$$

(5)

![Fig. 1. Local and global co-ordinate systems for 2-D continua](image)