Abstract. This paper shows a role of the contraction rule in decision problems for the logics weaker than the intuitionistic logic that are obtained by deleting some or all of structural rules. It is well-known that for such a predicate logic $L$, if $L$ does not have the contraction rule then it is decidable. In this paper, it will be shown first that the predicate logic $FLe_{ec}$ with the contraction and exchange rules, but without the weakening rule, is undecidable while the propositional fragment of $FLe_{ec}$ is decidable. On the other hand, it will be remarked that logics without the contraction rule are still decidable, if our language contains function symbols.

1. Introduction

In recent years, various studies have been done concerning logics lacking some or all of structural rules. (For more information see [13].) We will devote this paper to studying decision problems for some of these logics, which are weaker than the intuitionistic logic, and will show how the contraction rule plays a critical role in the decision problem for them. First, we will consider the logic $FLe_{ec}$ with both the contraction and exchange rules, but without the weakening rule. The logic $FLe_{ec}$ is no other than the intuitionistic relevant logic $R$ without distribution. The decidability of the propositional $FLe_{ec}$ will be shown by modifying a method originally developed for relevant logics. Next, the undecidability of the predicate $FLe_{ec}$ will be proved by reducing its decision problem to that of the intuitionistic predicate logic. On the other hand, if $L$ is one of these predicate logics and moreover if $L$ does not have the contraction rule, then it is decidable (see e.g. [8]). Using the idea introduced in [11], we will strengthen it and show that they are still decidable in the case where our language contains function symbols.

It is well-known that both the classical and the intuitionistic predicate logics are undecidable, while their propositional fragments are decidable. On the other hand, it was shown that predicate logics lacking some structural rules are decidable when they have no contraction rule. The standard way of showing that is to prove the cut elimination theorem in the first place. Then, it goes as follows. Suppose that the cut elimination theorem holds for a logic $L$ that does not have the contraction rule. Then, it can be shown that every cut-free proof has the property that each upper sequent of a given rule
of inference is simpler than its lower sequent. From this fact it follows that every decomposition-tree of a given sequent is finite, and that the number of its decomposition-trees is also finite. This gives us a procedure of deciding whether a given sequent is provable in L or not.

On the other hand, when a given logic L contains the contraction rule but not the weakening rule, some difficulties will occur, even if L is a cut-free propositional logic. Recall here that in the case of the classical and the intuitionistic propositional logics, the usual procedure relies on the fact that we can restrict the number of occurrences of a formula in a given sequent by virtue of the existence of both contraction and weakening rules. To overcome the difficulty, Kripke [9] introduced a method of proving decidability results, which was later extended to relevant logics by Belnap and Wallace [3] (see also [6]).

In §2, we will first prove the cut elimination theorem for FL_{ec}. Then by modifying Kripke's method, the decidability of the propositional FL_{ec} will be proved. In §3, we will give a translation of a given sequent of the intuitionistic predicate logic LJ into a sequent of the predicate FL_{ec} using the idea mentioned in the paper [13, §3] by the second author. Making use of this translation, we can reduce the decision problem of the predicate FL_{ec} to that of LJ, which is known to be undecidable.

The decision procedure for contraction-free logics mentioned in the above will have some difficulties when our language contains function symbols. For, a decomposition caused by quantification rules ($\rightarrow \exists$) and ($\forall \rightarrow$), i.e.

\[
\frac{\Gamma \rightarrow F(t)}{\Gamma \rightarrow \exists x F(x)} \quad \text{and} \quad \frac{F(t), \Gamma \rightarrow C}{\forall x F(x), \Gamma \rightarrow C},
\]

may produce the possibility of infinitely many upper sequents by the choice of the term $t$. (When we have no function symbols this difficulty can be avoided by taking the first variable not appearing in the lower sequent for $t$. See [8].) To overcome this, we will combine the decomposition procedure with the unification algorithm, by which we have the decidability results (see §4). We owe the idea to both Mey [11] and Voronkov (personal communication).

The authors would like to express their gratitude to Prof. S. Arikawa for his constant encouragement, Profs. M. Takano, A. Voronkov, K. Došen, and anonymous referee for their helpful comments.