M. J. CRESSWELL  Urn Models:  A Classical Exposition

Abstract. Urn models were developed by Veikko Rantala to provide a non-standard semantics for first-order logic in which the domains, over which the quantifiers range, are allowed to vary. Rantala uses game-theoretical semantics in his presentation, and the present paper is a study of urn models from a more classical, truth-conditional point of view. An axiomatic system for urn logic is set out and completeness is proved by the method of maximal consistent sets.

I once heard a paper (I think it must have been related to [4]) in which Philip Johnson-Laird was describing how people can make mistakes in reasoning. Suppose someone argues

\[ \begin{align*}
& (i) \text{ All } F \text{'s are } H \text{'s} \\
& (ii) \text{ All } G \text{'s are } H \text{'s} \\
& \therefore (iii) \text{ All } F \text{'s are } G \text{'s}
\end{align*} \]

This argument is of course notoriously invalid, and mostly we do not use it. When we do use it it is because something like this goes on: In our minds we try to build up the simplest possible model in which the premises are true, and see whether the conclusion is also true. If we are otherwise ignorant about \( F, G \) and \( H \) (as of course we never are) then the simplest model for (i) is a model in which there is only one thing and it is both \( F \) and \( H \), and the simplest model for (ii) is a model in which there is only one thing and it is both \( G \) and \( H \). So in the simplest models in which both (i) and (ii) are true, there is one thing and it is both \( F \) and \( G \). We do not usually choose such models because we usually know more about \( F, G \) and \( H \). But perhaps we do always do something akin to it and use, in our thinking, models of the world in which the domains of individuals are not as they are in reality. This is, in effect, the insight behind Veikko Rantala’s notion of an Urn Model. Rantala [6] based his work on that of Hintikka, and Hintikka [2] has championed the use of urn models in providing “impossible possible worlds” in order to throw some light on the problem of propositional attitudes.

Rantala’s work is done in the tradition of game-theoretical semantics [1, p. 100]. It is the aim of this paper to consider urn models from the point of view of classical truth-conditional model theory, and to provide a Henkin style completeness proof. Some work on urn models from a classical point of view has been carried out by Olin [5]. My paper will be
mildly autobiographical to begin with, in indicating how I came to make
sense for myself of urn models. I would like to thank Harry Lewis, who
explained them to me while he was visiting Wellington in July and August
of 1980.

1. Pre-urn models

I shall first set out the classical truth-conditional semantics for first-order
logic in a style that I have used in logic courses in Wellington.

We suppose we have a language $\mathcal{L}$ of LPC which contains individual
variables $x, y, z \ldots$ etc., predicate letters $\varphi, \psi, \chi \ldots$ etc., the logical symbols
\neg, \lor and parentheses. Parentheses enclosing a variable form a universal
quantifier, and formation rules and definitions are as usual.

An interpretation (where confusion might arise, we shall say more
specifically a classical interpretation) for $\mathcal{L}$ is a pair $<D, V>$ in which $D$

is a non-empty set (the domain of the interpretation) and $V$ a function
which assigns to every $n$-place predicate $\varphi$ a set $V(\varphi)$ of $n$-tuples from $D$.

Given $<D, V>$ there is a set $\mathcal{N}$ (strictly $\mathcal{N}_{(D,V)}$) of assignments to the
variables so that for every $v \in \mathcal{N}$, and every individual variable $x$, $v(x) \in D$.

We define what it is for a wff $a$ to be true in $<D, V>$ under an assignment $v$.

We write this as

$$<D, V> \models_v a$$

Sometimes the $<D, V>$ is omitted where no confusion can arise. '4' means

'not $\models$'.

The truth conditions of formulae of $\mathcal{L}$ under $<D, V>$ are:

If $a$ is atomic, say $\varphi(a_1, \ldots, a_n)$

then

$$\models_v a \iff \langle v(a_1), \ldots, v(a_n) \rangle \in V(\varphi)$$

$$\models_v \neg a \iff \not \models_v a$$

$$\models_v a \lor \beta \iff \models_v a \text{ or } \models_v \beta.$$  

We define $\mu \in \mathcal{N}$ to be an $x$-alternative of $v$, for any variable $x$, iff $\mu(y) = v(y)$ for every variable $y$ except (possibly) $x$.

$$\models_v (x) a \iff \models_\mu a \text{ for every } x\text{-alternative } \mu \text{ of } v.$$  

As usual $<D, V> \models a$ means that $<D, V> \models_v a$ for any $v \in \mathcal{N}_{(D,V)}$.

How then can we incorporate changing domains into predicate logic? The
simplest solution would be to modify $\mathcal{N}$ rather in the way Henkin
modified the model theory of higher order logic. I.e., instead of requiring
that $\mathcal{N}$ be all functions from individuals to members of $D$, why not
simply add $\mathcal{N}$ to the interpretation and let it specify which assign-
ments are to count? This is not quite what Rantala does but it

could be seen as on the way, so I shall briefly describe what I will call

a pre-urn model. The inadequacy of pre-urn semantics will provide a lead
in to urn models.