Abstract. The first known statements of the deduction theorems for the first-order predicate calculus and the classical sentential logic are due to Herbrand [8] and Tarski [14], respectively. The present paper contains an analysis of closure spaces associated with those sentential logics which admit various deduction theorems. For purely algebraic reasons it is convenient to view deduction theorems in a more general form: given a sentential logic $C$ (identified with a structural consequence operation) in a sentential language $\mathcal{S}$, a quite arbitrary set $P$ of formulas of $\mathcal{S}$ built up with at most two distinct sentential variables $p$ and $q$ is called a uniform deduction theorem scheme for $C$ if it satisfies the following condition: for every set $X$ of formulas of $\mathcal{S}$ and for any formulas $a$ and $\beta$, $\beta \in C(X \cup \{a\})$ iff $P(a, \beta) \subseteq C(X)$. [$P(a, \beta)$ denotes the set of formulas which result by the simultaneous substitution of $a$ for $p$ and $\beta$ for $q$ in all formulas in $P$.] The above definition encompasses many particular formulations of theorems considered in the literature to be deduction theorems. Theorem 1.3 gives necessary and sufficient conditions for a logic to have a uniform deduction theorem scheme. Then, given a sentential logic $C$ with a uniform deduction theorem scheme, the lattices of deductive filters on the algebras $\mathcal{A}$ similar to the language of $C$ are investigated. It is shown that the join-semilattice of finitely generated (= compact) deductive filters on each algebra $\mathcal{A}$ is dually Brouwerian. The crucial result of the paper — Theorem 2.11 — states that for a very wide class of logics the converse of the above result also holds. More specifically, if $C$ is a standard logic for which there exists a set $A(p, q)$ of sentential formulas built up with two variables such that $A(p, p) \subseteq C(0)$ and $q \in C(A(p, q) \cup \{p\})$ then $C$ has a uniform deduction theorem scheme iff for every algebra $\mathcal{A}$ similar to the language of $C$, the join-semilattice of compact deductive filters on $\mathcal{A}$ is dually Brouwerian.

§ 1. Uniform deduction theorem schemes

By a sentential logic we understand a pair

$$(\mathcal{S}, C),$$

where $\mathcal{S}$ is a sentential language, i.e., an absolutely free algebra freely generated by an infinite set $\text{Var}(\mathcal{S}) = \{p, q, r, \ldots\}$ of sentential variables and endowed with countably many finitary connectives $\&$, $\mid$, $\ldots$, and $C$ is a structural consequence operation on $\mathcal{S}$, the underlying set of the algebra $\mathcal{S}$. Thus $C$ satisfies the following conditions:

(a) $X \subseteq C(X)$, for all $X \subseteq \mathcal{S}$;
(b) if $X \subseteq Y$ then $C(X) \subseteq C(Y)$, for all $X, Y \subseteq \mathcal{S}$;
(c) $C(C(X)) = C(X)$, for all $X \subseteq \mathcal{S}$;
(d) $e(C(X)) \subseteq C(e(X))$, for every endomorphism $e$ of $\mathcal{S}$ and for all $X \subseteq \mathcal{S}$.

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The members of $S$ are called \textit{sentential formulas}. If $a$ is a formula, then $\text{Var}(a)$ denotes the set of sentential variables occurring in $a$. For any $X \subseteq S$ we set $\text{Var}(X) = \bigcup \{\text{Var}(a) : a \in X\}$. We shall often use the notation $X(p_1, \ldots, p_n)$ to indicate that $\text{Var}(X) = \{p_1, \ldots, p_n\}$. The endomorphisms of $S$ are customarily called \textit{substitutions} in $S$.

If no confusion is likely we shall identify a logic $(S, C)$ with its consequence operation $C$. Moreover, if $X$ is a finite set of formulas, then instead $C(X)$ we shall often write $C(a_1, \ldots, a_n)$, where $a_1, \ldots, a_n$ is a fixed arrangement of the elements of $X$. We also use $C(X, a)$ as an abbreviation for $C(X \cup \{a\})$.

A logic $C$ is \textit{standard} if

$$C(X) = \bigcup \{C(Y) : Y \subseteq X \& Y \text{ is finite}\},$$

for all $X \subseteq S$.

A logic $(S, C)$ is \textit{inconsistent} if $C(\emptyset) = S$. If the set $C(\emptyset)$ of theses of a logic $C$ is non-empty, then one easily verifies that $C$ is inconsistent iff $q \in C(p)$ for some distinct sentential variables $p$ and $q$.

A \textit{rule of inference} in a language $S$ is a subset $q \subseteq P(S) \times S$, where $P(S)$ is the power set of $S$. A rule $q$ is structural if it is invariant under substitutions of $S$, that is, if for each substitution $e$ in $S$ and any $X \subseteq S$, $a \in S$, $(eX, ea) \in q$ whenever $(X, a) \in q$. Given a pair $(X, a)$ from $P(S) \times S$ we define the structural rule $X/a$ to be the set $\{(eX, ea) : e \text{ is a substitution in } S\}$. The rules of the form $X/a$ are called \textit{sequential} and the pair $(X, a)$ is called a scheme of $X/a$. Notice that a scheme of $X/a$ is unique up to the choice of variables. A sequential rule $X/a$ is \textit{standard} when the set $X$ is finite. The standard rule $X/a$ is customarily denoted by $\gamma_1, \ldots, \gamma_k/a$, where $\gamma_1, \ldots, \gamma_k$ is a fixed arrangement of the elements of $X$. Modus Ponens $p, p \rightarrow q/q$ is an example of a standard rule. A standard rule $q$ is \textit{axiomatic} if it is of the form $\emptyset/a$.

We define the notion of a \textit{proof}. Let $\theta$ be a set of standard rules in $S$ and let $X \cup \{\beta\}$ be a set of formulas of $S$. A finite sequence of formulas

$$\beta_1, \ldots, \beta_n$$

is said to be a \textit{formal proof} of $\beta$ from $X$ by means of the rules from $\theta$ iff $\beta_n = \beta$ and for every $i$, $1 \leq i \leq n$, either $\beta_i \in X$ or there exist indices $i_1, \ldots, i_k < i$, a rule $\gamma_1, \ldots, \gamma_k/a$ in $\theta$ and a substitution $e$ in $S$ such that $\beta_{i_j} = e\gamma_{i_j}$ for $j = 1, \ldots, k$ and $\beta_i = ea$.

By $C_{\theta}(\beta)$ we denote the logic in $S$ determined as follows: $a \in C_\theta(X)$ iff there is a formal proof of $a$ from $X$ by means of the rules from $\theta$.

\footnote{In practice, when we deal with formalized logics (see Theorem 1.1 below) we shall not discern between the rule $\emptyset/a$ and the formula $a$ itself.}