Abstract. It is demonstrated how Kripke models for intuitionistic predicate logic can be applied in order to prove classical theorems. As examples proofs of the independence of the axiom of constructibility, of the omitting types theorem and of Shelah's ultrapower theorem are sketched.

Kripke models had been originally introduced as a semantics for intuitionistic and modal logics. Later it was realized by several people that they could be used as a tool in proofs of classical theorems. The aim of this paper is to present three different types of such applications. More precisely we shall sketch proofs of independence results in set theory, of the omitting types theorem and of the Shelah ultrapower theorem using Kripke models. Since we are here interested only in the role of Kripke models, our sketches will be far from being complete proofs.

Some knowledge of Kripke models and of the basic facts of modern model theory and set theory is presupposed. Unexplained notation is standard, except that we do not assume the standard interpretation of the equality symbol in Kripke models. Kripke models are denoted as \((\mathcal{U}, \mathcal{U})\), where \(\mathcal{U} = (\mathcal{A}, \preceq, P)\) is a partial ordering with the least element \(P\) and \(\mathcal{U}\) is a function assigning to each \(a \in \mathcal{A}\) a classical model \(\mathcal{U}_a\) such that for \(a \preceq b\) the identity is a homomorphic embedding of \(\mathcal{U}_a\) into \(\mathcal{U}_b\). If \(a \in \mathcal{A}\) and if \(\varphi\) is an atomic formula with parameters from \(\mathcal{U}_a\) (the universe of \(\mathcal{U}_a\)), then we say that \(a\) forces \(\varphi (a \models \varphi)\), if \(\mathcal{U}_a \models \varphi\). The forcing relation is extended to arbitrary first order formulas in the usual way (e.g. \(a \models \neg \varphi\) iff for all \(b \geq a\) non \(b \models \varphi\)). In some cases \(|\mathcal{U}_a|\) will be a proper class (not a set) but we shall not worry about such details. We say that \((\mathcal{U}, \mathcal{U})\) is a model of \(\varphi ((\mathcal{U}, \mathcal{U}) \models \varphi)\), if \(P \models \varphi\).

1. Independence results in set theory

As a typical example we discuss Cohen's proof of the relative consistency of the negation of the axiom of constructibility (i.e. the proof of \(\text{Con}(\text{ZFC}) \Rightarrow \text{Con}(\text{ZFC} + \forall V \neq L))\). The universe \(V\) of all sets can be build as follows. Let \(V_0 = \emptyset\). For every ordinal \(a\) let \(V_{a+1}\) be the powerset of \(V_a\) and if \(a\) is a limit ordinal let \(V_a = \bigcup_{\beta < a} V_\beta\). Then \(V\) is the union of all the sets \(V_a\) (\(a\) an ordinal). In a similar way a Kripke model of set theory will be built.
Let $A = \{a: a$ is a function, $dom(a)$ is a finite set of natural numbers and $rg(a) \subseteq \{0, 1\}\}$. Say $a \leq b$ iff $dom(a) \subseteq dom(b)$ and $a$ is the restriction of $b$ to $dom(a)$. Put $\mathfrak{U} = (A, \leq, \emptyset)$. For every $a \in A$ let $U^a_{\mathfrak{U}}$ be the empty structure for the language of set theory. Now suppose $U^a_{\mathfrak{U}}$ is defined for every $a \in A$ and for all such $a$ $|U^a_{\mathfrak{U}}| = U_a$. Let $W_a$ be the set of all functions $f$ such that $dom(f) \subseteq U_a$, $rg(f)$ is a subset of the powerset of $A$ and for all $t \in dom(f)$, $a \in f(t), b \in A$ $a \leq b$ implies $b \in f(t)$. For every $f \in W_a$ let $f^+$ be a new unary predicate symbol. Every model $U^a_{\mathfrak{U}}$ is expanded to a model $M^a_{\mathfrak{U}}$ of the new language such that $M^a_{\mathfrak{U}} \models f^+(t)$ iff $t \in dom(f)$ and $a \in f(t)$. By $\models$ we denote the forcing relation in the expanded model $(\mathfrak{U}, \mathfrak{M})$. Now the model $U^a_{\mathfrak{U}}$ is defined so that $|U^a_{\mathfrak{U}}| = U_a \cup W_a$ and $U^a_{\mathfrak{U}} \models f^+ \iff f \in W_a$ if one of the following conditions is satisfied.

1. $f, g \in U_a$ and $U^a_{\mathfrak{U}} \models f = g$
2. $f \in U_a, g \in W_a$ and $a \models f^+(f)$
3. $f \in W_a, g \in U_a$ and $a \models \exists t (t \in g \land \exists u \forall (f^+(u) \leftrightarrow u \in t))$
4. $f, g \in W_a$ and $a \models \exists t (f^+(t) \land \exists u \forall (f^+(u) \leftrightarrow u \in t))$.

If $a$ is a limit ordinal let $U^a_{\mathfrak{U}} = \bigcup_{\beta < a} U^\beta_{\mathfrak{U}}$. Finally let $U_a = \bigcup_{\alpha \in a} U^\alpha_{\mathfrak{U}}$. Equality is introduced so that $(\mathfrak{U}, \mathfrak{M}) \models \forall x \exists y (y = x \leftrightarrow \exists z \forall z (z \in x \leftrightarrow z \in y))$. It can be shown that for every axiom $\varphi$ of ZFC$(\mathfrak{U}, \mathfrak{M})$ is a model of a sentence which is classically equivalent to $\varphi$. For every set $c$ an element $c \in U$ is defined inductively so that $dom(c) = \{d: d \in c\}, c(d) = A$ for all $d \in c$. Then for all $c, d$

$$(\mathfrak{U}, \mathfrak{M}) \models c = d \iff c = d,$$

$$(\mathfrak{U}, \mathfrak{M}) \models \neg c = d \iff c \neq d,$$

$$(\mathfrak{U}, \mathfrak{M}) \models c \in d \iff c \in d,$$

$$(\mathfrak{U}, \mathfrak{M}) \models \neg c \in d \iff c \not\in d.$$}

In order to show that $V \neq L$ holds in $(\mathfrak{U}, \mathfrak{M})$ we have to find $a \in U$ such that $(\mathfrak{U}, \mathfrak{M}) \models \forall \neg g = c$ for all $c \in V$.

Let $dom(g) = \{\bar{n}: n \in \omega\}, g(\bar{n}) = \{a \in A: n \in dom(a) \text{ and } a(n) = 1\}$. Suppose $(\mathfrak{U}, \mathfrak{M})$ non $\models \neg g = c$. Then for some $a$ $a \models g = \check{c}$. Let $n \in \omega$ be such that $n \notin dom(a)$. Assume $n \notin c$. We can find some $b \supseteq a$ such that $n \in dom(b)$ and $b(n) = 1$. Then $b \models g = \check{c}$, hence $b \models \bar{n} \in g$. Since $b \models g = \check{c}$ we have $b \models \bar{n} \in c$, so $n \in c$ — a contradiction. If $n \in c$, we obtain a contradiction in a similar way.

So we have constructed a Kripke model of a theory which is classically equivalent to ZFC + $V \neq L$. Then there must be also a classical model of this theory.

Though it is not necessary for the proof, we indicate how such a model can be constructed.

If ZFC is consistent, there must be a countable model of ZFC. Let $\mathfrak{M}$ be such a model and suppose the construction of the Kripke model