The present paper is thought as a formal study of distributive closure systems which arise in the domain of sentential logics. Special stress is laid on the notion of a C-filter, playing the role analogous to that of a congruence in universal algebra. A sentential logic C is called filter distributive if the lattice of C-filters in every algebra similar to the language of C is distributive. Theorem IV.2 in Section IV gives a method of axiomatization of those filter distributive logics for which the class $\text{Matr}(C)_{\text{prime}}$ of C-prime matrices (models) is axiomatizable. In Section V, the attention is focused on axiomatic strengthenings of filter distributive logics. The theorems placed there may be regarded, to some extent, as the matrix counterparts of Baker's well-known theorem from universal algebra [9, § 62, Theorem 2].

§ I. Preliminaries

Our notation and terminology does not differ from that in common use. We also assume some acquaintance with basic notions and results of lattice theory. Thus by a sentential logic we mean a pair $(\mathcal{S}, C)$, where $\mathcal{S}$ is a sentential language, i.e., an absolutely free algebra freely generated by an infinite set $V(\mathcal{S})$ of sentential variables and endowed with countably many finitary connectives, $\&_1, \&_2, \ldots$, and where $C$ is a structural consequence on $S$, the set of formulas of $\mathcal{S}$. We usually identify a logic $(\mathcal{S}, C)$ with its consequence operation $C$. Thus a logic $C$ is finite iff $C(X) = \bigcup\{C(X_f): X_f$ is a finite subset of $X\}$, for all $X \subseteq S$. The endomorphisms of $\mathcal{S}$ are usually called substitutions in $\mathcal{S}$. Given a set $X$ of formulas of $\mathcal{S}$ we define $\text{Sb}(X)$ to be the set $\bigcup\{eX: e$ is a substitution in $\mathcal{S}\}$, where $eX = \{e\gamma: \gamma \in X\}$. $V(a)$ denotes the set of sentential variables which occur in the formula $a$ and for $X \subseteq S$ we write $V(X)$ to denote the set $\bigcup_{a \in X} V(a)$. A set $X \subseteq S$ is called a theory (or a system) of a logic $C$ if $C(X) = X$. The family of all theories of $C$ is denoted by $\text{Th}(C)$. We also set $C_1 \leq C_2$ to mean that a logic $C_2$ is a strengthening of $C_1$, i.e., $C_1(X) \subseteq C_2(X)$, for all $X \subseteq S$. It is easy to show that $C_1 \leq C_2$ iff $\text{Th}(C_2) \subseteq \text{Th}(C_1)$.

Rules of inference on $\mathcal{S}$ are relations between sets of formulas (premises) and formulas (conclusions); thus a rule on $\mathcal{S}$ is a subset of $\mathcal{P}(S) \times S$.

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where $P(S)$ is the power set of $S$. Given $X \subseteq S$ and $a \in S$ define

$$X/a = \{(eX, ea) : e \text{ is a substitution in } \mathcal{F}\}.$$ 

The rules of the form $X/a$ are called **sequential** and the pair $(X, a)$ is called a scheme of $X/a$. A sequential rule $X/a$ is **standard** when $X$ is finite and $X/a$ is **axiomatic** when $X$ is empty. $X/a$ is a **rule** of $C$ if $a \in C(X)$. Each set $Q$ of standard rules on $\mathcal{F}$ determines the finite logic $C_Q$ in $\mathcal{F}$:

$$a \in C_Q(X) \text{ iff there exists a finite sequence}$$

$$(1.1) \quad \gamma_1, \ldots, \gamma_n$$

of formulas such that $\gamma_n = a$ and for every $i, 1 \leq i \leq n$, either $\gamma_i \in X$ or there are indices $i_1, \ldots, i_k < i$ and a rule $\varrho \in Q$ such that the pair $\{(\gamma_{i_1}, \ldots, \gamma_{i_k}), \gamma_i\}$ belongs to $\varrho$.

$(1.1)$ is referred to as a **$Q$-proof** of $a$ from the set $X$. If $C$ is a finite logic, then by a **$C$-proof** we mean a proof carried out by means of any set of standard rules of $C$.

We shall devote much attention to axiomatic strengthenings of logics. Given a set $Q'$ of standard rules, it will be convenient for our purposes to discern in the set $Q'$ the axiomatic and the nonaxiomatic rules of inference. The set $Q'$ will be usually denoted as the join $A \cup Q$, where $A$ is the set of axiomatic rules in $Q'$ and $Q$ is the set of nonaxiomatic rules in $Q'$. We shall often identify an axiomatic rule $\Theta/a$ with the formula $a$. Then $C_{A \cup Q}(X) = C_Q(Sb(A) \cup X)$, for all $X \subseteq S$. We shall say that a logic $C'$ is an **axiomatic strengthening** of a logic $C$ iff there is a set $A$ of formulas such that $C'(X) = C(Sb(A) \cup X)$, for all $X \subseteq S$, i.e., iff $C'$ results from $C$ by adjoining to the rules of $C$ the set of rules $\Theta/a$, $a \in A$.

Given a logic $C$ we call a set $Q$ of standard rules an (inferential) **base** for $C$ if $C = C_Q$. Every finite logic has a base (see [10], cf. also [1]). A logic $C$ is **finitely based** if it has a finite base. Clearly every finitely based logic is finite.

A logical matrix is a pair $\mathcal{M} = (\mathcal{A}, D)$, where $\mathcal{A}$ is an abstract algebra, referred to as the algebra of $\mathcal{M}$, and $D$, the set of designated elements of $\mathcal{M}$, is a subset of $\mathcal{A}$, the underlying set of $\mathcal{A}$. $\mathcal{M}$ is called a **matrix for a language** $\mathcal{F}$ if the algebra of $\mathcal{M}$ is similar to $\mathcal{F}$. Any class $K$ of matrices for $\mathcal{F}$ is called a (matrix) **semantics** for $\mathcal{F}$. Every semantics $K$ for $\mathcal{F}$ induces the logic $Cn_K$ in $\mathcal{F}$, where $a \in Cn_K(X)$ iff for every matrix $\mathcal{M} = (\mathcal{A}, D) \in K$ and every homomorphism (valuation) $h : \mathcal{F} \rightarrow \mathcal{A}$, $ha \in D$ whenever $hX \subseteq D$. If $K$ consists of a single matrix $\mathcal{M}$ then the logic $Cn_K$ is denoted as $Cn_\mathcal{M}$. A semantics $K$ is **strongly adequate** for a logic $C$ if $C = Cn_K$. Given a logic $C$, we define

$$\text{Matr}(C)$$