A Semantical Investigation into
Leśniewski's Axiom of His Ontology

Abstract. A structure for the language \( L \), which is the first-order language (without equality) whose only nonlogical symbol is the binary predicate symbol \( \varepsilon \), is called a quasi \( \varepsilon \)-structure iff

(a) the universe \( |\mathcal{A}| \) of \( \mathcal{A} \) consists of sets and
(b) \( a \varepsilon b \) is true in \( \mathcal{A} \leftrightarrow (\exists p)[a = \{p\} \& p \in b] \) for every \( a \) and \( b \) in \( |\mathcal{A}| \), where \( a(b) \) is the name of \( a(b) \). A quasi \( \varepsilon \)-structure \( \mathcal{A} \) is called an \( \varepsilon \)-structure iff

(c) \( \{p\} \in |\mathcal{A}| \) whenever \( p \in a \in |\mathcal{A}| \). Then a closed formula \( \sigma \) in \( L \) is derivable from Leśniewski's axiom

\[ \forall x, y . [x \varepsilon y \rightarrow (\exists u)(u \varepsilon x) \land \forall u(v(u, v \varepsilon x \rightarrow u \varepsilon v) \land \forall u(u \varepsilon x \rightarrow u \varepsilon y))] \]

of his ontology, and a somewhat weaker axiom \( \mu \):

\[ \forall x, y . (x \varepsilon y \rightarrow x \varepsilon x) \land \forall x, y, z . (x \varepsilon y \land y \varepsilon z \rightarrow x \varepsilon z) \land \forall x, y, z . (x \varepsilon y \land y \varepsilon z \rightarrow y \varepsilon x) \]

which is indeed derivable from \( \lambda \) (Słupecki [4], T. 9.1, T. 7.1 and T. 10.1), and was studied in Ishimoto [1].

As in the usual model theory, a structure \( \mathcal{A} \) is a model of a closed formula \( \sigma \) iff \( \sigma \) is true in \( \mathcal{A} \). A structure \( \mathcal{A} \) is epimorphic to a structure \( \mathcal{B} \) iff an epimorphism (= onto homomorphism) of \( \mathcal{A} \) onto \( \mathcal{B} \) exists. A structure \( \mathcal{A} \) is a quasi \( \varepsilon \)-structure iff the universe \( |\mathcal{A}| \) of \( \mathcal{A} \) consists of sets and for every \( a \) and \( b \) in \( |\mathcal{A}| \),

\[ a \varepsilon \{b\} \text{ is true in } \mathcal{A} \leftrightarrow (\exists p)[a = \{p\} \& p \in b], \]

where \( a \) and \( b \) are the names of \( a \) and \( b \), respectively. A quasi \( \varepsilon \)-structure \( \mathcal{A} \) is an \( \varepsilon \)-structure iff \( \{p\} \in |\mathcal{A}| \) whenever \( p \in a \in |\mathcal{A}| \). Then we obtain the following Theorem and Corollary.

**Theorem 1°** A structure is a model of the axiom \( \lambda \) iff it is epimorphic to some \( \varepsilon \)-structure.

**Theorem 2°** A structure is a model of the axiom \( \mu \) iff it is epimorphic to some quasi \( \varepsilon \)-structure.
COROLLARY. 1°) A closed formula $\sigma$ is derivable from the axiom \( \lambda \) iff every $\varepsilon$-structure is a model of $\sigma$.

2°) A closed formula $\sigma$ is derivable from the axiom $\mu$ iff every quasi $\varepsilon$-structure is a model of $\sigma$.

Compare our corollary with Iwanuś' result:

IWANUŚ THEOREM (Iwanuś [2], Theorem 3.II). A closed formula $\sigma$ is provable in elementary ontology $EO$ iff every $\varepsilon$-structure whose universe is the power set of some set is a model of $\sigma$, where $EO$ is the theory which has $\lambda$ and the universal closure of each formula of the form $\exists x \forall y (u \in x \leftrightarrow u \in u \wedge \varphi)$, where $\varphi$ has no free occurrence of $x$, as nonlogical axioms.

In the body of this paper we shall correctly define each semantical notion along the line of Shoenfield [3], and then prove Theorem and Corollary.

Also following Shoenfield [3] we suppose that the logical symbols $\land$, $\rightarrow$, $\leftrightarrow$ and $\forall$ are defined in terms of $\neg$, $\lor$ and $\exists$.

1. Semantical preliminaries

Let $L$ be the first-order language (without equality) whose only non-logical symbol is the binary predicate symbol $\varepsilon$. Clearly the axioms $\lambda$ and $\mu$ which are mentioned in the introduction are (closed) formulas in $L$.

A structure $\mathcal{A}$ (for $L$) is a pair $<\mathcal{A}, \varepsilon_{\mathcal{A}}>$ of a nonempty set $\mathcal{A}$ and a binary predicate $\varepsilon_{\mathcal{A}}$ in $\mathcal{A}$.

Let $\mathcal{A} = <\mathcal{A}, \varepsilon_{\mathcal{A}}>$ be a structure. For each $a$ in $\mathcal{A}$, we choose a constant, called the name of $a$ and denoted by $a$. Let $L(\mathcal{A})$ be the first-order language obtained from $L$ by adding all the names of elements of $\mathcal{A}$. We shall define a truth value $\mathcal{A}(\tau)$ for each closed formula $\tau$ in $L(\mathcal{A})$ by induction on the length of $\tau$:

$$\mathcal{A}(a \in b) = T \iff \varepsilon_{\mathcal{A}}(a, b),$$
$$\mathcal{A}(\neg \tau') = T \iff \mathcal{A}(\tau') \neq T,$$
$$\mathcal{A}(\tau' \lor \tau'') = T \iff \mathcal{A}(\tau') = T \text{ or } \mathcal{A}(\tau'') = T,$$
$$\mathcal{A}(\exists x \tau') = T \iff \mathcal{A}(\tau'_{x}[a]) = T \text{ for some } a \text{ in } \mathcal{A}.$$

The structure $\mathcal{A}$ is a model of $\sigma$, which is a closed formula in $L$, iff $\mathcal{A}(\sigma) = T$. Hence, when $\mathcal{A}$ is a model of the axiom $\lambda$ or $\mu$,

$$\varepsilon_{\mathcal{A}}(a, b) \rightarrow \varepsilon_{\mathcal{A}}(a, a),$$
$$\varepsilon_{\mathcal{A}}(a, b) \& \varepsilon_{\mathcal{A}}(b, c) \rightarrow \varepsilon_{\mathcal{A}}(a, c),$$
$$\varepsilon_{\mathcal{A}}(a, b) \& \varepsilon_{\mathcal{A}}(b, c) \rightarrow \varepsilon_{\mathcal{A}}(b, a),$$

for every $a$, $b$ and $c$ in $\mathcal{A}$; which will be used later without mention.