A Note on Russell's Paradox in Locally Cartesian Closed Categories

Abstract. Working in the fragment of Martin-Löf's extensional type theory [12] which has products (but not sums) of dependent types, we consider two additional assumptions: firstly, that there are (strong) equality types; and secondly, that there is a type which is universal in the sense that terms of that type name all types, up to isomorphism. For such a type theory, we give a version of Russell's paradox showing that each type possesses a closed term and (hence) that all terms of each type are provably equal. We consider the kind of category theoretic structure which corresponds to this kind of type theory and obtain a categorical version of the paradox. A special case of this result is the degeneracy of a locally cartesian closed category with a morphism which is generic in the sense that every other morphism in the category can be obtained from it via pullback.

Introduction

At the heart of the categorical approach to logic lies the ability to identify certain kinds of logical theory with particular kinds of category-theoretic structure. Indeed, the search for such logic-category correspondences has formed a major part of the subject to date. Once achieved, such an identification has its uses in both directions: sometimes algebraic techniques can be applied to the category-theoretic structures to yield results about the logical theories; at other times proof-theoretic or model-theoretic results for the logic yield new properties of the categories involved. This note illustrates in a minor way this latter aspect of the subject. Consider the following category-theoretic result:

PROPOSITION. Let $C$ be a locally cartesian closed category (that is, $C$ has finite limits and for each object $X$ in $C$, the slice category $C/X$ is cartesian closed). Suppose further that $C$ contains a generic family, which by definition is a morphism $t: G \to U$ with the property that any other morphism $f: A \to X$ is a pullback of $t$ along some morphism $X \to U$. Then it is the case that $C$ is degenerate: every object in $C$ is isomorphic to the terminal object.

We will give a proof of this proposition which relies upon the correspondence established by Seely [14] between locally cartesian closed categories and theories over a (strong) version of Martin-Löf's system [12] of dependent types, with sums, products and equality types. Recall that under this correspondence the dependent types $A[x][x \in X]$ of a theory are modelled by morphisms $A \to X$ in the category and substitution is modelled by (chosen) pullback.

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operations (which we assume given when specifying the structure of a locally cartesian closed category). Then the assumption that \( C \) has a generic family \( t: G \to U \) translates into the assumption that the type theory possesses a universal type (or "type of all types") in the following sense:

There is a constant type \( \text{Type} \) whose terms are intended to name all the types, up to isomorphism; \( \text{Type} \) is modelled by the object \( U \) in \( C \). There is a dependent type \( T[u] \) giving the actual type named by \( u \in \text{Type} \); this is modelled by \( t: G \to U \) in \( C \). Whenever \( A \) is a type derivable in the theory, possibly dependent on some variables \( \bar{x} \) say (and modelled by \( A \to X \) in \( C \)), a constant \( n_A(\bar{x}) \in \text{Type} \) is introduced along with function constants and axioms to make \( A[\bar{x}] \) isomorphic to \( T[n_A(\bar{x})] \) (rather than actually equal to it). Since the result of substituting \( n_A(\bar{x}) \) for \( u \) in \( T[u] \) is modelled in \( C \) by forming the pullback of \( t: G \to U \) along the morphism \( f: X \to U \) modelling \( n_A(\bar{x}) \), the provable isomorphism \( A[\bar{x}] \cong T[n_A(\bar{x})] \) in the type theory means that in \( C \) the morphism \( A \to X \) is isomorphic to the (chosen) pullback of \( t \) along \( f \) (and hence is a pullback of \( t \) along \( f \)).

The proof of the proposition proceeds by giving a type-theoretic version of Russell's paradox. In this version, the roles of equality and universally quantified predicates are played as usual by equality and product types. The role of the powerset operation is played by the function type \((-) \to \text{Type} \); and the role of negation is played by the function type \((-) \to A \), where \( A \) is a fixed, but arbitrary type. In this form, the paradox produces (via an application of Cantor's familiar diagonal argument) a closed term of type \( A \). Since \( A \) was arbitrary, we conclude that all types possess closed terms; since in particular this can be applied to equality types, we conclude further that all terms of any particular type are provably equal. Back in the locally cartesian closed category, this means that all objects possess a global section and that all parallel pairs of morphisms are equal — from which degeneracy follows immediately.

The type-theoretic version of the paradox we present does not use dependent sums. (A slightly simpler argument could be given using sums.) As a result, we in fact prove a sharper result (Proposition 1.5) of which the above proposition is a special case.

The use we make of Russell's paradox in this paper should be contrasted with Girard's proof [6] of the inconsistency of Martin-Löf's original theory of types with a universal type, which used a type theory version of the Burali-Forti paradox. Recently there has been renewed interest from the computer science community in type theories with a universal type, but without equality types: see [2], [13] and [3]. Coquand [4] and Howe [7] analyse Girard's version of the paradox and show in particular that it can be carried out in type theory with just dependent products and a universal type. So such a type theory is "inconsistent" in the sense that every type possesses

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\(^2\) Added in proof: such an argument has been given independently by Boom [1]. We are grateful to A. S. Troelstra for bringing the existence of this preprint to our attention.