Graded Modalities. I*

Abstract. We study a modal system $\mathcal{T}$, that extends the classical (propositional) modal system $T$ and whose language is provided with modal operators $M_n (n \in \mathbb{N})$ to be interpreted, in the usual kripkean semantics, as "there are more than $n$ accessible worlds such that...". We find reasonable axioms for $\mathcal{T}$ and we prove for it completeness, compactness and decidability theorems.

1. Introduction

We study a modal system $\mathcal{T}$, that extends the classical (propositional) modal system $T$ (see e.g. [6]) and whose language is provided with modal operators $M_n (n \in \mathbb{N})$ to be interpreted, in the usual kripkean semantics, as "there are more than $n$ accessible worlds such that...". From this point of view, it is obvious that $M_0$ has the same interpretation that the classical possibility operator has. We find reasonable axioms for $\mathcal{T}$ and we prove for it completeness (via satisfiability), compactness and decidability theorems.

Of course, the graded modalities $M_n$ (with their dual ones, $L_n$) can be inserted in every modal system with a kripkean semantics: we present here the case of $\mathcal{T}$, extension of $T$, just as an example of a trend that can be followed for other classical systems, like $K$, $B$, $S4$, $S5$, etc., so to generate the corresponding extensions $\mathcal{K}$, $\mathcal{B}$, $\mathcal{S4}$, $\mathcal{S5}$, etc.

We observe also that the idea of grading the classical modalities $M$ and $L$ is not entirely new in itself: Goble [5] introduces grades for modal operators, but in an entirely different conceptual framework and with different technical features.

2. The syntax of $\mathcal{T}$

The language of $\mathcal{T}$ consists of:

a) a denumerable set of basic propositional symbols, $\mathcal{P} = \{p, q, r, \ldots\}$, with or without indexes from $\mathbb{N}$ (the set of natural numbers);

b) the propositional connectives $\neg$ and $\vee$;

c) the modal operators $M_n (n \in \mathbb{N})$;

d) a finite set of parentheses.

* The authors are very indebted to the referee for his consideration and appreciation of their work.
The set of (well-formed) formulas of $\mathcal{T}$, $(\mathcal{S})$, is defined as usual except for allowing such formulas as $M_n\alpha$ if $\alpha$ is a formula. Formulas will be denoted by small Greek letters, with or without indexes. We shall write $\lor\{a_i: \ldots i \ldots \}$, where $\ldots i \ldots$ is some condition on $i$, to indicate a finite but long disjunction, and similarly for conjunctions.

We adopt the usual convention to introduce the remaining propositional connectives as abbreviations of formulas built only by $\rightarrow$ and $\lor$, as well as the following modalities:

- **A1)** classical propositional tautologies
- **A2)** $a \rightarrow M_0\alpha$
- **A3)** $M_{n+1}\alpha \rightarrow M_n\alpha$ (n $\in N$)
- **A4)** $L_0(a \rightarrow \beta) \rightarrow (M_n\alpha \rightarrow M_n\beta)$ (n $\in N$)
- **A5)** $M^*_0(a \land \beta) \rightarrow ((M^*_n\alpha \land M^*_n\beta) \rightarrow M^*_{n+m}(a \lor \beta))$ (n, m $\in N$).

The **rules of inference** of $\mathcal{T}$ are:

- **(MP)** From $\alpha, \alpha \rightarrow \beta$ derive $\beta$
- **(N)** From $\alpha$ derive $L_0\alpha$.

The notions of a formal deduction and of a theorem are defined in usual way: $\vdash \alpha$ means "$\alpha$ is a theorem (of $\mathcal{T}$)".

**Theorem 1.** a) If $\vdash \alpha \rightarrow \beta$ then $\vdash M_n\alpha \rightarrow M_n\beta$ (n $\in N$).

b) If $\vdash \alpha \leftrightarrow \beta$ then $\vdash M_n\alpha \leftrightarrow M_n\beta$ (n $\in N$).

c) If $\alpha, \beta \in (\mathcal{S})$ and $\beta$ differs from $\alpha$ for having a formula $\delta$ in some (not necessarily all) places in which a formula $\gamma$ occurs in $\alpha$, and $\vdash \gamma \leftrightarrow \delta$, then $\vdash \alpha \leftrightarrow \beta$.

**Proof.** a) and b) are proved by easy verifications; c) is proved by a simple induction on complexity of formulas, using tautologies and b).

Theorem 1 allows the introduction of some derived rules of inference for $\mathcal{T}$:

- **(DR1)** From $\alpha \rightarrow \beta$ derive $M_n\alpha \rightarrow M_n\beta$ (n $\in N$)
- **(DR2)** From $\alpha \leftrightarrow \beta$ derive $M_n\alpha \leftrightarrow M_n\beta$ (n $\in N$)
- **(Eq)** The usual rule of substitution of equivalent formulas (see e.g. [6]).