OSCILLATING FLOWS OF MISCIBLE FLUIDS

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Abstract
A theoretical analysis is presented for the flows of two miscible, viscous, incompressible fluids, subject to oscillatory pressure gradients in a cylindrical tube. The extended, time-dependent Navier-Stokes equations with inter-component interaction for the fluids are reduced to ordinary, inhomogeneous differential equations of fourth order by separation and decoupling, and solved in closed form. As distinguished from the classical one-fluid Richardson flow, the dynamics of the two-component, oscillatory flow is determined by two eigenvalues which are complex. Since the coupling force between the fluids increases with their velocity difference, deviations from the one-fluid Richardson effect exist.

§ 1. Introduction
In contrast to steady-state Hagen-Poiseuille flow, the oscillatory flow of a viscous, incompressible fluid in a tube exhibits a velocity maximum close to the walls if the (oscillatory) Reynolds number \( Re = \omega R^2/\nu \) is large. This effect was first observed by Richardson [1] who measured the mean square of the oscillatory fluid velocity to have a maximum close to the wall rather than at the center of the pipe. This oscillatory velocity distribution has been explained theoretically in complete agreement with the experimental findings [2, 3]. Recently, the theory of the Richardson effect has been extended to particle suspensions in viscous fluids [4, 5], where the particle velocity \( v_p \) is coupled to the fluid velocity \( v \) through a phenomenological (non-viscous) relaxation equation of the form \( \partial v_p/\partial t = -(v_p - v)/\tau. \)

In the following, we analyse the oscillating flows of a mixture of two different, viscous, incompressible fluids, which are subject to sinusoidally

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varying pressure gradients in a cylindrical tube. Each of the miscible fluids is described by an extended Navier-Stokes equation, in which the dynamic coupling of the different fluids is due to intercomponent friction forces which satisfy the action-reaction principle. These partial differential equations of second order are reduced to two ordinary, inhomogeneous differential equations of fourth order by separation and decoupling. The latter differential equations are solved in closed form in terms of the modified Bessel functions $I_0(k_{\pm}r)$, where $k_+$ and $k_-$ are complex eigenvalues. The method of solution is applicable to many-fluid mixtures, where the resulting ordinary differential equations are of order $2^n$ if $n$ is the number of fluids in the mixture.

In general, the radial amplitude and phase distributions of the velocity fluctuations of the miscible fluids are different depending on their densities $\rho_s$, viscosities $\mu_s$ and relaxation times $\tau_s, s = 1, 2$. The oscillatory motions of the interacting fluids are determined by a novel oscillatory Reynolds number $Re_m = \omega R^2/v_m, v_m = v_m(v_1, v_2)$. The theory presented is applicable to miscible fluids of the Navier-Stokes type, e.g. C$_2$H$_5$OH and H$_2$O, as well as to liquid metal mixtures, e.g. Na and K (in absence of electric and magnetic fields). It is also applicable to binary gas mixtures, e.g. A and Ne, presuming that the oscillatory gas motions are subsonic, since gases are quasi-incompressible for subsonic flow conditions. In a dilute gas mixture, the relaxation times $\tau_s$ are large, and the fluids oscillate increasingly uncoupled with increasing dilution.

The results might be applicable to dilute particle suspension in viscous fluids, but only as much as the mean oscillatory motion of the suspended particles is concerned. However, the concept of two classical, interacting, Navier-Stokes fluids is, in general, not applicable to a dilute suspension of macroscopic particles in an incompressible viscous fluid, since the spatial distribution of the suspended particles is non-uniform. For example, in blood flow there is a lower number density of red blood cells close to the wall than in the centre [6]. If the viscosity of one of the fluids is set zero, our results reduce to those of the relaxation equation approach [4, 5]. It should be noted, however, that the suspended fluid exhibits a viscosity even if it is highly diluted as a result of cross-collisions between the particles of the suspended and background fluids [7, 8, 9].

§ 2. Boundary-value problem

Under consideration are two miscible, viscous, incompressible fluids of mass density $\rho_s$, viscosity $\mu_s$, and momentum relaxation time $\tau_s$ in a quasi-