Maximizing a Correlational Ratio for Linear Extensions of Posets

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Abstract. Let \( \rho = P(12)/P(12|13) \), where \( P(ij) \) is the probability that \( i \) precedes \( j \) in a randomly chosen linear extension of a partially ordered set \( (\{1, 2, \ldots, n\}, \leq) \) in which points 1, 2 and 3 are mutually incomparable. A previous paper by the author (Order 1, 127 (1984)) proved that \( \rho < 1 \). The present paper considers the maximization of \( \rho \) for each \( n > 3 \). It shows that, with \( \alpha_n = \lfloor (n+3)/2 \rfloor \), the maximum \( \rho \) is at least

\[
\left[ \frac{\alpha_n n}{\alpha_n n} - n \right] / \left[ \frac{\alpha_n n}{\alpha_n n} - \alpha_n \right].
\]

Evidence that this value cannot be exceeded is given. It is also proved that the smallest possible value of \( P(231) + P(321) \) is

\[
\frac{1}{\lfloor (n+1)/2 \rfloor}.
\]


Key words. Partial order, linear extension, correlation.

1. Introduction

Throughout, \( n > 3 \), \( n = \{1, 2, \ldots, n\} \), \( \leq \) is an asymmetric and transitive binary relation on \( n \), and \( i \sim j \) if neither \( i < j \) nor \( j < i \). The largest integer not exceeding \( x \) is \( \lfloor x \rfloor \).

Let \( \mathcal{R}_n \) be the family of posets \( (n, \leq) \) for which \( 1 \sim 2 \sim 3 \sim 1 \). Also let \( P(ij) \) denote the probability that \( i \) precedes \( j \) in a randomly chosen linear extension of a given poset in \( \mathcal{R}_n \), so \( P(ij) = 1 \Leftrightarrow i < j \). By [1], \( P(12) < P(12|13) \), the strict version of what has been called the \( xyz \) inequality [3]. Following a suggestion of Rival [2], we consider

\[
\rho_n = \max_{\mathcal{R}_n} \frac{P(12)}{P(12|13)}
\]
Lacking an exact solution for $\rho_n$, we derive fairly tight bounds on this maximum correlational ratio and suggest that the lower bound might actually be the exact solution.

*Given a poset in $\mathcal{P}_n$, let $\rho = P(12)/P(12\mid 13)$, let $N$ be the number of linear extensions of the poset, and let $N(i_1, i_2, \ldots, i_k)$ be the number of these $N$ in which $i_1$ precedes $i_2$ ... precedes $i_k$. Our approach to $\rho_n$ is based on the fact that $\rho = 1 - (1 - \lambda)/K_0$, where

$$
\lambda = \frac{N(213)N(312)}{[N(123)+N(132)][N(231)+N(321)]}
$$

and

$$
K_0 = \frac{N}{N(231)+N(321)}.
$$

The maximum value of $\lambda$ of $(n-1)^2/(n+1)^2$ for odd $n$ and $(n-2)/(n+2)$ for even $n$ was established in [1]. We prove here that the maximum value of $K_0$ is $\frac{\lfloor n+1/2 \rfloor}{\lfloor n+1/2 \rfloor}$. Our upper bound on $\rho_n$ is simply obtained by inserting these maxima in the preceding expression for $\rho$.

The upper bound gives the exact value of $\rho_n$ for $n \in \{3, 4\}$ since the poset configurations that maximize $\lambda$ and $K_0$ coincide for these $n$. However, this is not true for $n \geq 5$. Our lower bound on $\rho_n$ for these cases is based on the fact that $1/K_0$ tends to be much smaller than $1 - \lambda$ for values near the maxima. We therefore focus on posets that tend to maximize $K_0$ in deriving a good lower bound.

The next section presents our main results and some numerical calculations. Configurations that maximize $K_0$ are established in the third section. The final section offers subsidiary results which suggest that $\rho_n$ might in fact equal our lower bound.

2. Main Results

**Theorem 1.** For each $n \geq 3$,

$$
K_0 \leq \left\lfloor \frac{n+1}{2} \right\rfloor
$$

for all posets in $\mathcal{P}_n$, with equality holding if and only if the poset satisfies the restrictions of Figure 1.

Theorem 1 is proved in the next section.

On Figure 1, $a \sim b$ whenever $a \in A \cup \{1\}$ and $b \in B \cup \{2, 3\}$. The $<$ arrangement of points within $A$ and within $B$ does not affect $K_0$. Indeed, if $(A \cup \{1\}, <)$ has $N_1$ linear extensions (1 always at the bottom, or leftmost) and $(B \cup \{2, 3\}, <)$ has $N_2$