ON THE NOETHER MAP

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ABSTRACT. For a wide class of Lagrangian systems we show rigorously that the conventional formulation of Noether's theorem provides a bijective map from the set of equivalence classes of Noether's symmetries onto the set of equivalence classes of conserved currents. We further discuss if Noether's theorem is generalized in a significant way by several formulations proposed in this decade.

1. INTRODUCTION

In a recent paper of Ibragimov [1] a method is proposed to relate conserved currents with symmetries for arbitrary systems of partial differential equations by means of a previously formulated generalization of Noether's theorem [2]. On the other hand, Candotti-Palmieri-Vitale [3, 4] and Rosen [5, 6] have generalized Noether's theorem in such a way that conserved currents appear in general to be unrelated with symmetries even for Lagrangian systems (see in particular [6] and Theorem 1 of [4]). Then the following questions naturally arise:

(1) Does Noether's theorem provide a deep relationship between symmetries and conservation laws for Lagrangian systems?

(2) To what extent is it possible to generalize Noether's theorem?

The present note is devoted to investigate these two questions. It is formulated in terms of algebraic methods as introduced by Gel'fand and Dikii in [7], and it uses several notions and results of a previous paper [8]. In order to keep the paper within usual language we adopt the formal term 'infinitesimal transformation' to describe the vector fields of first-order differential operators acting on the algebra of regular density functions.

Our analysis is based on the concept of an integrating factor of a system of partial differential equations, which is closely related to the conserved currents of the first kind [9]. It allows us to isolate that property of Lagrangian systems which is implicit in the conventional formulation of Noether's theorem [10]. This property is that integrating factors of Lagrangian systems define symmetries. It leads in a natural way to the Noether map between Noether symmetries and conservation laws. Our main task is to deduce rigorously under what conditions this map is bijective. The answer to this question must take into account two aspects. First, the algebraic

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nature of the discussion requires obtaining an algebraic characterization of the ideal of density functions which vanish on the solutions of the field equations; and, secondly, there are natural notions of equivalence in the sets of symmetries and conserved currents which suggest that it is more intrinsic to consider the Noether map in terms of equivalence classes. In this way, we are able to prove rigorously the bijective character of the Noether map for normal Lagrangian systems. Finally, we conclude that the above-mentioned generalizations of Noether’s theorem do not improve in any sense the deep property of Lagrangian systems provided by the conventional formulation of Noether’s theorem.

2. FORMULATION OF THE RESULTS

We follow the notation conventions of [8]. Let \( R \) be the algebra of \( C^\infty \) functions \( F = F[x, u] \) depending upon \( n \) independent variables \( x_i (i = 1, \ldots, n) \) and derivatives of arbitrary order of \( m \) dependent variables \( u^r (r = 1, \ldots, m) \). By a normal system of partial differential equations we shall mean a system of the form

\[
\frac{\partial^{N_{r+1}}}{\partial t^{N_{r+1}}} F[x, u] = 0, \quad r = 1, \ldots, m, \tag{1}
\]

where \( t \) denotes one of the coordinates \( x_i \), \( N_r \) are non-negative integers, and \( u \) denotes the set of variables \( u^r = (u_1, \ldots, u_m) \) such that the component of \( u \) corresponding to the coordinate \( t \) is less than or equal to \( N_r \). Clearly, if \( \theta^r \in R \) all the variables \( u^r \) may be written as \( C^\infty \) functions of the variables \( x, u, \omega \), where \( \omega \) denotes the set of functions \( D^\alpha \omega^r \) \((|\alpha| \geq 0, r = 1, \ldots, m)\). Then, every element of \( R \) may be expressed as a \( C^\infty \) function \( F[x, u, \omega] \). Let \( I \) be the ideal of functions \( F \in R \) which vanish when the field equations \( (1) \) are satisfied. That is, \( F \in I \) if and only if \( F[x, u, 0] = 0 \). The following lemma establishes an important property of normal systems.

**Lemma 1.** The ideal \( I \) is generated by the functions \( D^\alpha \omega^r \) \((|\alpha| \geq 0, r = 1, \ldots, m)\).

**Proof.** Evidently \( D^\alpha \omega^r \in I \). On the other hand, from a fundamental theorem of calculus we have

\[
F[x, u, \omega] = F[x, u, 0] + \sum_{r, \alpha} D^\alpha \omega^r \int_0^1 \frac{\partial F}{\partial D^\alpha \omega^r} \circ \gamma(\tau) d\tau
\]

where \( \gamma \) is the curve \( \gamma(\tau) = [x, u, \tau \omega] \). The conclusion follows at once. \( \square \)

Given \( F, G \in R \) we write \( F \equiv G \) when \( F - G \in I \). A vector function \( \vec{A} = (A_1, \ldots, A_m) (A_i \in R) \) is said to be a conserved current of \( (1) \) if \( \vec{D} \cdot \vec{A} \equiv 0 \). Clearly, if \( \vec{A} \equiv 0 \) or \( \vec{D} \cdot \vec{A} \equiv 0 \), then \( \vec{A} \) is a conserved current. Two conserved currents \( \vec{A} \) and \( \vec{A}' \) are said to be equivalent if \( \vec{A} - \vec{A}' = \vec{B} + \vec{C} \) with \( \vec{B} \equiv 0 \) and \( \vec{D} \cdot \vec{C} \equiv 0 \). From Lemma 1 every conserved current of \( (1) \) must satisfy an equation of the form \( \vec{D} \cdot \vec{A} = \sum_{r, \alpha} \lambda^r_{\alpha} \cdot D^\alpha \omega^r \) where \( \lambda^r_{\alpha} \in R \) and the sum extends to a finite number of terms.

An \( m \)-component function \( \lambda = (\lambda^1, \ldots, \lambda^m) \) is said to be an integrating factor of \( (1) \) if the following system of equations is satisfied

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