Exact Solutions of Nonrelativistic Classical and Quantum Field Theory with Harmonic Forces

To the memory of Marek Kac and Stanislaw Marcin Ulam - great scientists and our fine friends

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Abstract. Nonlinear field equations describing nonrelativistic particles interacting via harmonic forces are exactly solved. Collective variables of the system are identified and their evolution is found. Full quantum solution of the initial value problem is exhibited and compared with the classical solution of the same problem.

Harmonic oscillators have frequently served as the best laboratory for theoretical physicists. Using harmonic oscillators one may construct models at all levels of complexity: from a single classical oscillator in one dimension to relativistic quantum fields.

In this Letter I shall use the field-theoretic description of a system of harmonic oscillators to study the relationship between the quantum and the classical theories. This relationship is very simple in the standard approach to the N-oscillator problem, since the classical solution of the equations of motion can also be directly used in the quantum case, due to the linearity of the equations. The linearity of the equations of motion, however, is completely lost in the formulation of the many-oscillator problem in terms of the annihilation and creation operators.

The equation of motion for the field operator $\Psi(x, t)$ annihilating an oscillator at the point $x$ has the form

$$i\hbar \partial_t \Psi(x, t) = \left[ \frac{p^2}{2m} + \frac{m \omega^2}{2} \int dx' \Psi^\dagger(x', t) \Psi(x', t)(x' - x)^2 \right] \Psi(x, t), \quad (1)$$

where

$$p = -i\hbar \partial_x$$

and it has the Hermitian conjugate form for the field operator $\Psi^\dagger(x, t)$ creating an oscillator at $x$. The nonlinearity of these equations is due to a greater generality of the description as compared to the standard approach; the field equations fully describe the system made of an unspecified number of oscillators. This nonlinearity is a source of interesting differences between the classical and quantum descriptions.
In order to simplify the formulas I shall choose the units of length and time that are built from the three constants appearing in the problem. This choice results in the replacements

$$x \rightarrow (\hbar/m\omega)^{1/2} x, \quad t \rightarrow t/\omega, \quad \Psi \rightarrow (\hbar/m\omega)^{1/4} \Psi$$

and the three constants drop out of all the equations.

The field equation (1) and the total Hamiltonian of the system in the dimensionless form are:

$$i \partial_t \Psi(x, t) = \left[\frac{p^2}{2} + N(x - X(t))N^2/2 + U(t)\right] \Psi(x, t),$$

$$H = p^2/2N + T(t) + NU(t),$$

where I have introduced the following set of operators representing the global, collective variables

$$N = \int dx \, \Psi^\dagger \Psi,$$

$$X(t) = \int dx \, x \Psi^\dagger \Psi,$$

$$P = \int dx \, \Psi^\dagger p \Psi,$$

$$T(t) = \frac{1}{2} \int dx \, (p \Psi^\dagger p \Psi - \frac{1}{2N} P^2),$$

$$U(t) = \frac{1}{2} \int dx \, \Psi^\dagger xx^2 \Psi - \frac{1}{2N} X^2(t),$$

$$W(t) = \frac{1}{4} \int dx \, \Psi^\dagger (xp + px) \Psi - \frac{1}{4N} (X(t)P + PX(t)).$$

These operators will prove very useful in the forthcoming analysis. They form a closed commutator algebra, because their only nonvanishing equal-time commutators are

$$[X, P] = iN, \quad [T, U] = -2iW,$$


The fact that the $X$ and $P$ operators commute with the remaining global operators indicates that the center of mass motion has been completely separated from the quadrupole oscillations described by the operators $T, U$ and $W$. These last three operators obey the commutation relations of the generators of the Lorentz group in $2 + 1$