On the Dimension of Some Modular Irreducible Representations of the Symmetric Group

OLIVIER MATHIEU
Department of Mathematics, Institute of Advanced Mathematics Research, Louis Pasteur University,
7 rue Rene Descartes, F-67084 Strasbourg Cedex, France. e-mail: mathieu@math.u-strasbg.fr

(Received: 5 August 1995)

Abstract. We compute the dimension of some irreducible representations of the symmetric groups in characteristic \( p \) (Theorem 2). The representations considered here are associated with Young diagrams \( m_1 \geq m_2 \geq \cdots \geq m_k \) such that \( m_1 - m_k \leq (p - 1) \). The formula is based on a variant of Verlinde's formula which computes some tensor product multiplicities of indecomposable modules for \( \text{GL}_N(\mathbb{F}_p) \).


Key words: Young diagrams, Symmetric groups, Frobenius formula, Verlinde's formula.

Introduction

In this Letter, we will compute the dimension of some modular irreducible representations of the symmetric group \( \Sigma_N \), (see Theorem 2 below for a precise statement). By a classical formula of Frobenius, the dimension of a characteristic zero irreducible \( \Sigma_N \)-representation is given as the number of standard tableaux of a given shape. However in the modular case, it is not very convenient to use the standard tableaux to describe these dimensions. Instead, we will use a combinatorial description based on paths in the set of Young diagrams. For this reason, we will first ‘translate’ the classical Frobenius formula in terms of paths.

\[ \begin{array}{c|cccc}
    & 1 & 2 & 3 & 4 \\
\hline
1 & X & X & X & X \\
2 & X & X & X & X \\
3 & X & X & X & X \\
4 & X & X & X & X \\
\end{array} \]

\text{Fig. 1}.
Recall that a Young diagram of height \( \leq l \) is a sequence of nonnegative integers \( \mathbf{m} = m_1 \geq m_2 \geq \cdots \geq m_l \). Pictorially one represents a Young diagram as in Figure 1, namely as a set of boxes with \( m_1 \) boxes on the first line, \( m_2 \) boxes on the second line and so on. The total number \( m_1 + m_2 + \cdots \) of boxes will be called the size of the Young diagram \( \mathbf{m} \). In order to give a completely rigorous definition, we also require that two Young diagrams which can be obtained one from the other by adding or removing empty lines, are considered as identical. For example, the Young diagrams \( 3 \geq 1 \) and \( 3 \geq 1 \geq 0 \) are viewed as the same.

![Figure 1]

Let \( Y_l \) be the set of all Young diagrams of height \( \leq l \). We consider \( Y_l \) as an oriented graph. Actually, there is an oriented edge going from \( \mathbf{m} \) to \( \mathbf{m}' \) if and only if we have \( m'_i = m_i \) for all indices \( i \) except for one, say \( j \), for which we have \( m'_j = m_j + 1 \). Pictorially, this means that we can get \( \mathbf{m}' \) from \( \mathbf{m} \) by adding exactly one box to \( \mathbf{m} \), (Figure 2). Denote by \( \emptyset \) the Young diagram with no boxes. To each Young diagram \( \mathbf{m} \) of size \( N \), Frobenius associated an irreducible \( \mathbb{C} \)-representation \( E\mathbb{C}(\mathbf{m}) \) of \( \Sigma_N \) and he proved the following result.

**THEOREM 1 (Frobenius formula in terms of paths).** The dimension of the complex representation \( E\mathbb{C}(\mathbf{m}) \) is the number of oriented paths from \( \emptyset \) to \( \mathbf{m} \).

Actually, the Frobenius Theorem was stated in terms of tableaux of shape \( \mathbf{m} \). We recall that a standard tableau of shape \( \mathbf{m} \) is a one-to-one labeling of the \( N \) boxes of \( \mathbf{m} \) by the integers \( 1, 2, \ldots, N \) which is increasing along the lines and the columns. Actually, it is easy to define a bijection between standard tableaux of shape \( \mathbf{m} \) and paths from \( \emptyset \) to \( \mathbf{m} \). Given a standard tableau of shape \( \mathbf{m} \), one can associate a path \( \emptyset = \tau_0, \tau_1, \ldots, \tau_N = \mathbf{m} \) going from \( \emptyset \) to \( \mathbf{m} \) with the requirement that \( \tau_k \) is the Young tableau of all boxes with label \( \leq k \). Conversely, one obtains a standard tableau from a path \( \emptyset = \tau_0, \tau_1, \ldots, \tau_N = \mathbf{m} \) by labeling with \( k \) the unique box of \( \tau_k \setminus \tau_{k-1} \).

Now fix a prime number \( p \) and two positive integers \( l \) and \( N \). Set \( k = \mathbb{F}_p \). By using the Schur Weyl duality, one can associate to any Young diagram \( \mathbf{m} \) of size \( N \) a \( k \)-representation \( E_k(\mathbf{m}) \) of \( \Sigma_N \). These representations \( E_k(\mathbf{m}) \) are irreducible or \( \{0\} \), and the nonzero representations \( E_k(\mathbf{m}) \) form a complete set of irreducible representations of \( \Sigma_N \) (see Section 3 for more details).

Let \( Y_l(p) \) be the set of all Young diagrams \( \mathbf{m} = m_1, \ldots, m_l \) of height \( \leq l \) such that \( m_1 - m_l \leq p - l \). We will prove the following theorem.