WEYL QUANTIZATION AND METAPLECTIC REPRESENTATION

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ABSTRACT. The formal expansion defining the twisted exponential of an element of the Lie algebra $\mathscr{H}_n \square (\oplus^n\ \mathscr{S}p(2, \mathbb{R}))$ can be summed and this result is used to explicitly obtain the classical function u_t corresponding to an evolution operator associated to a quantum Hamiltonian belonging to the above mentioned Lie algebra.

Then, by applying the Weyl quantization procedure to u_t we get a representation of the group $W_n \square (\oplus^n\ \mathrm{Sp}(2, \mathbb{R}))$ in terms of integral operators, the kernels of which are expressed by means of the classical action. The family u_t being only locally defined, it must be considered as a distribution on the classical phase space in order to get the metaplectic representation.

In a recent work [1] the classical functions u_t on the classical phase space, associated to the quantum evolution operator U_t corresponding to some linear physical systems have been calculated by using a trace formula.

By introducing the notion of twisted exponential [2, 3] these classical functions u_t can be directly calculated from the corresponding classical Hamiltonians. In the first part of this paper the functions u_t associated to Hamiltonians belonging to the algebra $\mathscr{H}_n \square \{\oplus^n\ \mathscr{S}p(2, \mathbb{R})\}$ are given explicitly.

The evolution operators of the considered systems being one-parameter subgroups of the inhomogeneous symplectic group, the metaplectic representation should be obtained by quantization of the classical functions u_t. But we shall see, in the second part of this paper, that the lack of continuity of u_t prevents us from obtaining a continuous representation and that it is necessary to consider u_t as a distribution on the phase space to construct the metaplectic representation.

I. Let Z denote a classical phase space, i.e. an even dimensional vector space over reals $Z \simeq \mathbb{R}^{2n}$ $(0 < n < \infty)$ with a fixed symplectic form.

It is well known that the Weyl correspondence rule Ω is not an homomorphism neither between the product of classical functions and the product of the corresponding quantum operators, nor between the Lie algebra structure induced on $C^\infty(Z)$ by the Poisson bracket $P(.,.)$ and the one defined by the usual commutator of the corresponding quantum operators.

The notion of twisted product has been introduced [5a] to make the Weyl correspondence rule an algebraic homomorphism for the product. The twisted product of two classical observables

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f and g is defined by the following formal series

$$f \Box g = fg + \sum_{m=1}^{\infty} (\tfrac{1}{2}\, i\hbar)^m (m!)^{-1} P^m(f, g), \tag{1}$$

where $P^m(.,.)$ denotes the mth-power of the Poisson bracket. This bilinear differential operator on $C^\infty(Z) \times C^\infty(Z)$ to $C^\infty(Z)$ can be expressed by

$$P^m(f, g) = f(\overleftarrow{\nabla} \cdot J \overrightarrow{\nabla})^m g, \tag{2}$$

where
- $\overrightarrow{\nabla} f$ is the gradient of f,
- $f \overleftarrow{\nabla}$ its transpose,
- J is the matrix corresponding to the symplectic form of Z.

Formally one can write

$$f \Box g = \exp(\tfrac{1}{2}\, i\hbar\, P)\,(f, g) \tag{3}$$

and for such a product one gets the desired property

$$\Omega(f \Box g) = \Omega(f)\Omega(g). \tag{4}$$

In a similar way it has been shown that the second above mentioned problem is solved by introducing a twisted Poisson bracket denoted $\mathscr{P}$, which is constructed as follows by using the non-commutativity of the twisted product

$$\mathscr{P}(f, g) = f \Box g - g \Box f. \tag{5}$$

The symmetry properties of the powers of the Poisson bracket

$$P^m(f, g) = (-1)^m P^m(g, f) \tag{6}$$

allow to formally write the twisted Poisson bracket as

$$\mathscr{P}(f, g) = 2\hbar^{-1} \sin(\tfrac{1}{2}\hbar P)\,(f, g). \tag{7}$$

Under this form we recognize the bracket introduced in Reference 5 which corresponds to the quantum commutator since

$$\Omega(\mathscr{P}(f, g)) = (i\hbar)^{-1} [\Omega(f), \Omega(g)]\,. \tag{8}$$

Relations (4) and (8) express the fact that the Weyl quantization rule becomes an algebraic homomorphism on a suitable function space endowed with the twisted product and the twisted Poisson bracket.