CARTAN'S GEOMETRICAL STRUCTURE OF SUPERGRAVITY*

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*To be submitted for publication.
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ABSTRACT. We emphasise the geometrical partnership of the vierbein and the spin-$\frac{3}{2}$ field in the structure of the supergravity Lagrangian. Both fields are introduced as components of the same matrix differential form. The only local symmetry of the theory is $SL(2, C)$.

Recently we have expressed [1] the supergravity Lagrangian [2] in Cartan's geometrical language [3] of differential forms. The spin-$\frac{3}{2}$ field is introduced [1] through a spinor differential one-form. Our purpose in this note is to stress the geometrical partnership of the vierbein field and the spin-$\frac{3}{2}$ field. This is done by introducing both fields through the same matrix differential one-form.

We start by defining the canonical vierbein matrix form

\[ V = \begin{pmatrix} \gamma_{a} e^{\mu}_{a} & \psi_{\mu} \\ 0 & 0 \end{pmatrix} \, dx^{\mu}. \tag{1} \]

This is an incomplete $OSP(4,1)$ matrix [4]. We use this matrix without requiring any algebraic generalization of the original $SL(2, C)$ structure [5] of the Einstein-Cartan theory of gravity.

We introduce the Lorentz connection form

\[ B = \begin{pmatrix} \frac{1}{4} g_{ab} B_{\mu ab} \\ 0 \end{pmatrix} \, dx^{\mu}. \tag{2} \]

The curvature two-form and the torsion two-form are defined, respectively, by the following $SL(2, C)$ covariant differentials:

\[ R = (d + iB) \wedge B, \tag{3} \]

\[ T = (d + iB) \wedge V. \tag{4} \]

From Equations (1)-(4) we find

\[ R = \begin{pmatrix} \frac{1}{4} g_{ab} R_{\mu ab} \\ 0 \end{pmatrix} \, dx^{\mu} \wedge dx^{\nu}, \tag{5} \]

Letters in Mathematical Physics 2 (1977) 115-117. All Rights Reserved.
\[ T = \begin{vmatrix} \frac{1}{2} \gamma_i T_{\mu \nu a} & W_{\mu \nu} \\ 0 & 0 \end{vmatrix} \quad \text{d}x^\mu \wedge \text{d}x^\nu. \] (6)

Where we have the curvature and torsion components

\[ R_{\mu \nu a b} = \partial_\mu B_{\nu a b} + B_{\mu a c} B_{\nu c b} - (\mu \leftrightarrow \nu), \] (7)

\[ T_{\mu \nu a} = \partial_\mu e_{\nu a} + B_{\mu a b} e_{\nu b} - (\mu \leftrightarrow \nu), \] (8)

\[ W_{\mu \nu} = \partial_\mu \psi_{\nu} - \frac{i}{2} B_{\mu a b} \sigma_{a b} \psi_{\nu}. \] (9)

Now, as in the usual \( SL(2, C) \) theory \[5b\], we construct

\[ V \wedge V = \begin{vmatrix} -\frac{i}{2} \sigma_{a b} e_{\mu a} e_{\nu b} & \frac{1}{2} e_{\mu a} \gamma_5 \psi_{\nu} \\ 0 & 0 \end{vmatrix} \quad \text{d}x^\mu \wedge \text{d}x^\nu. \] (10)

We define the dual of (10) in the following manner:

\[ * (V \wedge V) = \begin{vmatrix} -\frac{i}{2} \sigma_{a b} \gamma_5 e_{\mu a} e_{\nu b} & 0 \\ \frac{1}{2} \bar{\psi}_{\mu} \gamma_5 e_{\nu a} & 0 \end{vmatrix} \quad \text{d}x^\mu \wedge \text{d}x^\nu, \] (11)

and we construct the Lagrangian density

\[ \mathcal{L} = \text{Tr} * (V \wedge V) \wedge \{ R + T \} \]

\[ + \text{Tr} * (V \wedge V) \wedge (V \wedge V). \] (12)

Substituting Equations (5), (6) and (11) and integrating, we obtain the usual action of supergravity \[1\], \[2\]. The second term of (12) contains both the mass term of the spin-\( \frac{3}{2} \) field and a cosmological form.

It is clear from the above construction that the Einstein-Cartan \( SL(2, C) \) framework of gravity does not admit local supersymmetry. One might attempt \[6\], however, generalizing that framework into \( OSP(4,1) \), so that the vierbein \( e_{\mu}^a \), the spin-\( \frac{3}{2} \) field \( \psi_{\mu}^a \) and the Lorentz gauge field \( B_{\mu}^{a b} \) are treated on an equal footing as three independent gauge fields. In that case the structure of the Lagrangian is completely different from supergravity. Yet one might ask what is behind the divergence cancellations that are noted \[7\] in the higher-order computations of the theory. From our point of view, it could be this geometrical partnership between the vierbein and the spin-\( \frac{3}{2} \) field described above. We hope to be able to exploit this geometrical structure in order to shed more light on the quantization problem.