CONSTRUCTION OF TWISTED PRODUCTS FOR COTANGENT BUNDLES OF CLASSICAL GROUPS AND STIEFEL MANIFOLDS

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ABSTRACT. The existence of invariant twisted products (deformations of the associative algebra of \( C^\infty \)-functions) on the cotangent bundles of classical groups and Stiefel manifolds is proved by explicit constructions. All these products are positive.

On a symplectic manifold \( W \), we can define Classical Mechanics by means of two algebraic structures defined on the vector space \( N = C^\infty (W; R) \)

1. a structure of associative algebra defined by the usual product of functions.
2. a structure of Lie algebra defined by the Poisson bracket \( \mathcal{P} \).

We have shown ([1], [2]) that it is reasonable to study if we can obtain by suitable deformations of these two structures a model isomorphic to the Quantum Mechanics. An important theorem of J. Vey [5] proves the existence of suitable deformations of the Poisson Lie algebra. The general problem of the existence of suitable associative (non commutative) deformations of the associative algebra \( N \) (or \( \ast \)-products for \( N \)) appears as much more difficult. Some procedures for defining such deformations has been given [1] in the context of the coadjoint representations of Lie algebras.

We prove here the existence of good invariant \( \ast \)-products for the symplectic manifolds defined by the cotangent bundles of classical groups and Stiefel manifolds. A study concerning the Grassmann manifolds is given.

For the construction, we use for one part procedures of quotient and product, and for another part invariant 'pseudo-metrics'.

1. NOTION OF \( \ast \)-PRODUCT

The definitions and notations are those of [1]. In particular \( Q^r \) is a bidifferential operator on \( N \) of maximum order \( r \) \((r > 1)\) in each argument, null on the constants, such that the principal symbol of \( Q^r \) coincides with the principal symbol of \( \mathcal{P} \) (for an arbitrary symplectic connection \( \Gamma \)). We take \( Q^0 (u, v) = uv, Q^1 (u, v) = \mathcal{P}(u, v) \). We suppose that \( Q^r \) is symmetric in \((u, v)\) if \( r \) is even, skewsymmetric if \( r \) is odd (condition of symmetry).

DEFINITION A. \( \ast \)-product for the symplectic manifold \((W, \Lambda)\) is defined by a bilinear map:

\[
N \times N \rightarrow E(N; \ast ; v)
\]

given by:

\[
Q^r (u, v) = \mathcal{P}(u, v)
\]

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\[ u \ast v = \sum_{r=0}^{\infty} (v' / r') \, Q^r(u, v) \quad (u, v \in N). \quad (1.1) \]

and satisfying the associativity relation which can be translated by:

\[
\sum_{r+s=t} (1/r! s!) \, Q^r(u, v), w) = \sum_{r+s=t} (1/r! s!) \, Q^r(u, Q^s(v, w))
\]

\[ (r, s \geq 0; t = 1, 2, ...). \quad (1.2) \]

2. TWISTED PRODUCT AND PRODUCT OF SYMPLECTIC MANIFOLDS

(a) Choose on the symplectic manifold \((W, \Lambda)\) a symplectic connection \(\Gamma\); \(Q^r\) can be written on the domain \(U\) of a chart \(\{x^I\}\):

\[ Q^r(u, v)_{\{x^I\}} = \sum_I A^I_{(t)} \, \nabla_{t_1} \ldots \nabla_{t_r} u \, \nabla_{i_1} \ldots i_{t'} I \, v \quad (t, t' \leq r), \]

where the coefficients \(A^I_{(t)}\) are symmetric with respect to the \(i\)'s, symmetric with respect to the \(j\)'s and define a tensor \(A_{(t)}\). We associate to \(Q^r\) the polynomial.

\[ \Pi^r(\xi, \eta) = \sum_I A^I_{(t)} \, \xi_{i_1} \ldots i_{t'} I \, \eta_{j_1} \ldots j_{t''} I. \quad (2.1) \]

(b) Consider two symplectic manifolds \((W_1, \Lambda_1), (W_2, \Lambda_2)\) and the product manifolds \((W_1 \times W_2, \Lambda_1 + \Lambda_2) = (W, \Lambda)\). Introduce auxiliary symplectic connections \(\Gamma_1\) and \(\Gamma_2\) on the factors; we obtain a symplectic connection on \((W, \Lambda)\). Suppose that \((W_1, \Lambda_1)\) (resp. \((W_2, \Lambda_2)\)) admits a \(\ast^1\)-product (resp. a \(\ast^2\)-product) corresponding to the bidifferential operators \(Q^r\) (resp. \(Q^s\)) on \(N(W_1) = N_1\) (resp. \(N(W_2) = N_2\)).

If \(\{x^\alpha\}\) (resp. \(\{x^\lambda\}\)) is a chart of \((W_1, \Lambda_1)\) of domain \(U_1\) (resp. \((W_2, \Lambda_2)\) of domain \(U_2\)) we have with evident notations:

\[ \Pi^1_1(\xi_1, \eta_1) = \sum_I A^I_{(t)} \, \xi_{\alpha_1} \ldots \alpha_{t'} I \, \eta_{\beta_1} \ldots \beta_{t''} I \]

and

\[ \Pi^2_2(\xi_2, \eta_2) = \sum_J B^J_{(t)} \, \xi_{\lambda_1} \ldots \lambda_{t'} J \, \eta_{\mu_1} \ldots \mu_{t''} J. \]

We obtain, by product of these polynomials, a polynomial defining on \(N(W) = N\) a bidifferential operator denoted by \(Q^r_1 Q^s_2\).

We set on \(N\):

\[ u \ast v = \sum_{r=0}^{\infty} (v' / r!) \, Q^r(u, v) \quad (u, v \in N), \]