SOME ESSENTIALLY SELF-ADJOINT DIRAC TYPE OPERATORS, II
(RESOLVENT ESTIMATES NEAR INFINITY)

J.J. LANDGREN
School of Mathematics, Georgia Institute of Technology, Atlanta, Georgia 30332,
U.S.A.

ABSTRACT. We obtain norm estimates for the resolvent of the Dirac operator. These estimates verify the key assumptions of an extended version of the Rellich-Kato theorem concerning essential self-adjointness. The application of this abstract theorem then shows that the one-electron Dirac operator can admit a Coulomb potential, for atomic numbers less than or equal to 118, plus a second perturbation which satisfies a mild Stummel type bound, and remain essentially self-adjoint.

4. INTRODUCTION

In this second part of this paper we supply the estimates needed to prove Theorem 2.1 which was stated in the first part. Specifically we verify the assumptions of the abstract Theorem 3.1 for the case of the Dirac operator described in Section 2. We then derive Theorem 2.1 from this application of Theorem 3.1. For convenience we continue the numbering of sections, however we start anew the numbering of references.

In Section 6, we verify the assumptions of Theorem 3.1 under the simplifying assumption that the potential $p$ is spherically symmetric. The general case requires further estimates and has been shown elsewhere [9.a]. This simplifying assumption allows us to make essential use of the fact that each one-electron Dirac operator admits a complete family of reducing subspaces. Applying Theorem 3.1 to each reducing subspace we find that assumptions (3.2) and (3.3) are the most difficult to verify. We accomplish this in Theorem 5.1, and proof of which is divided into five lemmas. In Lemma 5.1 we specify an explicit formula for the kernel of the part of the operator $\left(\hat{\mu} - H(0)\right)^{-1}$ over a reducing subspace. We then use the Carleman norm [7] in Lemma 5.2 to estimate the norm of $V_0(e)\left(\hat{\mu} - H(0)\right)^{-1}$. In Lemma 5.3 we formulate abstract criteria for the invertibility of an operator. In the next two lemmas we verify these criteria for the part of the operator $I - V_0(e)\left(\hat{\mu} - H(0)\right)^{-1}$ over a reducing subspace. In Lemma 5.4 we first define a family of operators which are compact modulo the operator $I - V_0(e)\left(\hat{\mu} - H(0)\right)^{-1}$. Then in Lemma 5.5 we show that the spectral radius of these operators is less than one and hence they are invertible. This in turn implies the desired existence and boundedness of the part of the operator $\left[ I - V_0(e)\left(\hat{\mu} - H(0)\right)^{-1}\right]^{-1}$ over a reducing subspace.
5. PROOF OF THEOREM 2.1

We prove Theorem 2.1 under the simplifying assumption that the potential $p_1$ is spherically symmetric. As is well known [6], the operator $H_0(e)$ admits a complete family of reducing subspaces. In view of our simplifying assumption, the operator $H_1(e)$ has the same reducing subspaces.

To describe the parts of $H_0(e)$ over these subspaces we let $D$ denote the differentiation operator and $M$ the multiplication operator on $C_0^\infty(0, \infty)$, the class of infinitely differentiable functions whose support is a compact and proper subset of $(0, \infty)$. Then set

$$M^{-1}f(\xi) = \frac{1}{\xi}f(\xi), \quad f \in C_0^\infty(0, \infty)$$

and for each real number $e$ define the family of operators

$$L(e)(\kappa) = \begin{pmatrix} (-D + \kappa M^{-1})f_2 + (I - eM^1)f_1 \\ (D + \kappa M^{-1})f_1 - (I + eM^1)f_2 \end{pmatrix} \text{ on } C_0^\infty((0, \infty), C_2), \quad (5.1)$$

$$e \in \mathbb{R}, \text{ and } \kappa = \pm 1, \pm 2, \ldots$$

Then it is shown that [3] [6], the part of $H_0(e)$ over each of its reducing subspaces is unitarily equivalent to an operator of the form of definition (5.1). It follows from the definition of $V_1$, that the parts of $H_1(e)$ over each reducing subspace are unitarily equivalent to $L(e)(\kappa) + M(p_1)$. Since the orthogonal sum of essentially self-adjoint operators is essentially self-adjoint it suffices to show that $H_1(e)$ is essentially self-adjoint on each reducing subspace.

In the abstract Theorem 3.1, we thus let

$$A = L(0)(\kappa), \quad V_0 = M^{-1}, \quad V_1 = M(p_1) \quad (5.2)$$

and

$$\mathcal{H} = L_2((0, \infty), C_2), \quad \mathcal{D} = C_0^\infty((0, \infty), C_2).$$

It is not difficult to show [9.b], that assumptions (3.1); and (3.4) of Theorem 3.1 hold for the operators and spaces just described. In the following theorem we verify assumptions (3.2) and (3.3). For convenience we define

$$R_\mu(\kappa)(0) = (i\mu - L(0)(\kappa))^{-1}, \quad \mu \neq 0, \quad \mu \in \mathbb{R}. \quad (5.3)$$

THEOREM 5.1 For each real $\kappa$ and for each $e \in \left( -\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2} \right)$ there exists a positive constant $\mu_0$ such that for $|\mu| > \mu_0$