ALL LOCAL CLASSICAL SYMMETRIES IN HAMILTONIAN MECHANICS

E. AGUIRRE*, H.D. DOEBNER and J.D. HENNIG
Institute for Theoretical Physics, Technical University of Clausthal, D-3392 Clausthal, F.R. Germany

ABSTRACT. Using the formalism of symplectic group actions and coadjoint orbits, we give a complete list of all classical simple Lie algebras which are local symmetries for a given Hamiltonian vector field.

1. INTRODUCTION

Consider a mechanical system and its Hamiltonian vector field $X$ on the phase space $(M, \omega_M)$, i.e., on a $2n$-dimensional manifold $M$ with symplectic structure $\omega_M$. For fixed $m \in M$ a (symmetry) algebra of $m$-local integrals of $X$ is a set $\mathfrak{g}$ of functions $f$ on $M$ such that on a neighborhood of $m$

(i) $Xf = 0, \ f \in \mathfrak{g}$,

(ii) $\mathfrak{g}$ is a Lie subalgebra of the Poisson algebra on $(M, \omega_M)$. The connection between $X$ and its local integrals, depending on $m$, is an obvious physical question and a topic of standard textbooks on mechanics. The following answer, generally not mentioned in textbooks, is quite simple and somewhat surprising from a physical point of view:

(A) Up to isomorphisms, the local symmetry algebras of $X$ only depend on the dimension $2n$ of $M$ (and not on $X$ and $m \in M$, resp.).

A result of this type is already encoded in the work of Lie [1] and is stated in Eisenhart [2]. The argument behind (A) is the observation that different Hamiltonians on $M$ are (locally) related through canonical transformations and that the Poisson brackets are invariant under these transformations (Section 2). The above result leaves the following problem open: Which $\mathfrak{g}$ are (up to isomorphisms) local symmetry algebras for the given dimension $2n$ of $M$? Some partial answers are known for simple $\mathfrak{g}$. We quote as examples the result [3] that all real forms of the complex Lie algebras $\mathfrak{sl}(k, \mathbb{C}) (k \geq 2)$, so $(k, \mathbb{C} (4 \neq k \geq 3), \mathfrak{sp}(k/2, \mathbb{C}) (k \geq 2, \text{even})$ are local symmetry algebras for $k \leq n - 1$, and we mention the conjecture [4] that semisimple $\mathfrak{g}$ with rank $\mathfrak{g} > n - 1$ are forbidden (for simple $\mathfrak{g}$, a proof based on function group methods [2] was given in [5]).

* Present address: Departamento de Mecánica, Facultad de Matemáticas, Universidad Complutense, Madrid-3, Spain.
** Manifolds and mappings between manifolds are taken here from the $C^\infty$-category.

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A complete answer for simple Lie algebras is presented in this letter. The appropriate mathematical framework is the theory of coadjoint orbits. It is straightforward to show (Section 3) that any simple \( g \) can be realized as a local symmetry algebra in \((M, \omega_M)\), iff \( d_{\min}(g) \leq 2n - 2 \), with \( d_{\min}(g) \) being the minimal dimension of all nontrivial coadjoint orbits of \( g \) in \( g^* \). In the case of classical simple Lie algebras, the results of Wolf [6] are used to determine a complete list of all such \( g \) (Section 4).

2. LOCAL SYMMETRIES OF HAMILTONIAN VECTOR FIELDS

We are interested in 'local' properties and 'local' objects on \((M, \omega_M)\). Hence, the formalism of germs is technically useful (although not necessary) and mathematically justified, because realizations of simple Lie algebras \( g \) by local vector fields on a neighborhood of \( m \in M \) are possible iff there are corresponding realizations of \( g \) by vector-field germs in \( m \) [5].

Denote by \( X(m) \), \( T_x(m) \), and \( \tau(m) \), resp. the set of function germs \( f \) (vector field germs \( X \) and \( (r, s) \) tensor-field germs \( \tau \), resp.) in \( m \) and by \( f \in \mathcal{F}(m) \), \( X \in \mathcal{F}(m) \), ..., the 'representative' functions, vector fields, etc. The linear and differential operations for germs in \( m \) are well defined via operations for corresponding representatives. Hence, \( X(m) \), \( T_x(m) \) are modules over the ring \( \mathcal{F}(m) \). Moreover, \( \mathcal{F}(m) \) is an infinite dimensional Lie algebra with respect to the Lie bracket of vector fields.

Consider now \((M, \omega_M)\) and the germ \( \mu \in T_x(m) \) of co\( \omega_m \), \( m \in M \) fixed. We define the subalgebra \( X_{\mu}(m) \subset X(m) \) of Hamiltonian vector field germs in \( m \) ('generators of local canonical transformations') by

\[
X \in X_{\mu}(m) := \{ f \in \mathcal{F}(m) \mid df = 0 \}
\]

and the set of \( m \)-local symmetries of \( X \) by

\[
S^X_{\mu}(m) := \{ f \in X_{\mu}(m) \mid [X, f] = 0 \}
\]

The Poisson bracket of functions induces a Lie algebra structure on \( \mathcal{F}(m) \) (\( \{ f, g \} := \mu(\tau(f, X_g)) \)) and \( I^X(m) \) turns out to be a subalgebra. Similarly, the Jacobi identity for germs yields \( S^X_{\mu}(m) \) as an infinite dimensional subalgebra of \( X_{\mu}(m) \). Thus, \( \tau : I^X(m) \to S^X_{\mu}(m), f \to X_f \), is a Lie algebra antihomomorphism, \( [\tau(f), \tau(g)] = -[X_f, X_g] \).

Given an abstract simple Lie algebra \( g \) and assuming \( X \in X_{\mu}(m) \) to be regular \( (X_m \neq 0 \) for \( X \in \mathcal{F}(m), i.e., m \) is not an equilibrium point of the system), we look for necessary and sufficient conditions to realize \( g \) by local integrals of \( X \) and/or by local symmetries of \( X \), i.e., conditions for the existence of injective homomorphisms \( g \to I^X(m) \) and/or \( g \to S^X_{\mu}(m) \). Because the second