A DUALITY BETWEEN MASSLESS PARTICLES AND
A CHARGE-MONOPOLE SYSTEM

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ABSTRACT. It is pointed out that the same realisation of a unitary representation of SO(4, 2)
from the ladder series is used in the coordinate-space description of a charge-monopole system,
(or more generally a dyon-dyon system) and in the canonical momentum-space description of a
massless particle. Therefore, in the latter case a momentum-space analogue appears for the
monopole vector potential, complete with its Dirac string singularity. Analogues of gauge trans-
formations relate equivalent realisations with different locations of the momentum-space string.
Quantisation of helicity replaces quantisation for the product of electric and magnetic charge.
The problem of localising a charge on a monopole string is related to recent work by Flato et al.
[5] on the localisability of a massless particle in momentum-space. Further, the multi-component
form of the generators of SO(4, 2) for a massless particle has a dual, in coordinate space,
corresponding to a charge-monopole system for a monopole of the Yang–Mills type. The question
of the interpretation of the momentum analogue of the monopole field is raised.

In ‘canonical’ momentum space realisations [1] of a zero mass, fixed helicity, unitary representation
of the Poincaré group, a line singularity appears in the expressions for some of the generators, as
derived by Shirokov [2], Lomont and Moses [3], and others [4].

The angular momentum operators in particular have the form

\begin{align}
J_1 &= -i(p_2 \nabla_3 - p_3 \nabla_2) + ip p_1 / \left[ p^2 - (p_3)^2 \right] \\
J_2 &= -i(p_3 \nabla_1 - p_1 \nabla_3) + ip p_2 / \left[ p^2 - (p_3)^2 \right] \\
J_3 &= -i(p_1 \nabla_2 - p_2 \nabla_1)
\end{align}

in Shirokov’s description, and
\[
J_1 = -i(p_2 \bar{\nabla}_3 - p_3 \bar{\nabla}_2) + j
\]  
\[\text{(2a)}\]
\[
J_2 = -i(p_3 \bar{\nabla}_1 - p_1 \bar{\nabla}_3) + j\bar{p}_2 / [p + p_1]
\]  
\[\text{(2b)}\]
\[
J_3 = -i(p_1 \bar{\nabla}_2 - p_2 \bar{\nabla}_1) + j\bar{p}_3 / [p + p_1]
\]  
\[\text{(2c)}\]
in the description given by Lomont and Moses. (In each case \(\bar{\nabla}_i = \partial / \partial p_i, p = [(p_1)^2 + (p_2)^2 + (p_3)^2]^{1/2}\) and \(j\) is the helicity, a fixed integer or half-integer.) The singular line in the former case is the whole \(p_3\)-axis; in the latter case, it is the nonpositive part of the \(p_1\)-axis.

Naive manipulation with these singular expressions can lead to apparently paradoxical results. For example, if we consider the Shirokov forms (1), suppose that the massless particle has a precise momentum \(\lambda\) directed along the \(3\)-axis, and assume that the state vector \(\psi(p)\) accordingly has the form
\[
\psi(p) \sim \delta(p_1) \delta(p_2) \delta(p_3 - \lambda),
\]  
\[\text{(3)}\]
then it appears that \(J_3\) vanishes and that
\[
J \cdot p = J_2 p_3 = 0,
\]  
\[\text{(4)}\]
contradicting the defining relation
\[
J \cdot p = j p
\]  
\[\text{(5)}\]
which should hold in this representation. Similarly, if we consider the Lomont–Moses forms (2) and suppose
\[
\psi(p) \sim \delta(p_1 - \lambda) \delta(p_2) \delta(p_3),
\]  
\[\text{(6)}\]
then it appears that \(J_1\) has the value \(j\) and that
\[
J \cdot p = J_1 p_1 = j \lambda,
\]  
\[\text{(7)}\]
which again contradicts (5) if \(\lambda\) is negative, since \(p\) has the value \(|\lambda|\). Such apparent paradoxes have been pointed out recently by Flato et al. [5], who have gone on to discuss some implications for the localisability of massless particles in momentum space. They emphasize that, for the Lomont–Moses forms, the correct space of (generalized) state vectors contains \(\psi\) as in (6) only if \(\lambda\) is positive. When \(\lambda\) is negative, they indicate the need to introduce instead a “twisted \(\delta\)-function” carrying angular momentum \(-2j\) in order to describe correctly the state of the particle, while maintaining the forms (2) for the angular momentum operators. They suggest that the subtleties associated with the definition of sharp momentum states may have previously unrecognised implications in theories involving massless particles.

In this note we point out that the situation is completely analogous to another one in which