WAVE OPERATORS FOR THE SCHRODINGER EQUATION WITH STRONGLY SINGULAR SHORT-RANGE POTENTIALS

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ABSTRACT. We use a semigroup positivity preserving to prove asymptotic completeness of the wave operators in many cases when they exist.

1. INTRODUCTION AND BASIC RESULTS

The potential scattering theory for Schrödinger operators \( H = H_0 + q \) where \( H_0 = -\Delta \) and \( q \) is a multiplication by function \( q(x) : \mathbb{R}^l \to \mathbb{R}^l \) acting in \( \mathcal{H} = L^2(\mathbb{R}^l) \) has been developed into very satisfactory manner in the last decade. In 1967 Kupsch and Sandhas [1] have shown the wave operators

\[
\Omega_\pm(H, H_0) = \lim_{t \to \pm \infty} e^{itH} e^{-itH_0}
\]

to exist for the large class of short-range interactions. Thus the following conclusions have been derived

\[
R(\Omega_\pm) \subset \mathcal{H}_{ac}(H) \subset \mathcal{H}_p(H)^\perp \subset R(E_H((0, \infty))).
\]

Here \( R(A) \) denotes the range of an operator \( A \), \( \mathcal{H}_p(H) \) is the subspace of \( \mathcal{H} \) generated by all the eigenvectors of \( H \), \( \mathcal{H}_{ac}(H) \) is the subspace of absolute continuity of \( H \) and \( E_H((a, b)) \) is the spectral projector for \( H \) corresponding to the set \( (a, b) \subset \mathbb{R}^l \).

The simplest problem of scattering theory is in proving the equality

\[
R(\Omega_\pm) = \mathcal{H}_{ac}(H).
\]

We shall denote this type of asymptotic completeness of wave operators by (KC). Most general results about (KC) have been obtained by Kato and Kuroda [2, 3] and Simon [4]. The main tool of their method is the theorem of Birman [5]. Another approach has been developed by Lavine [6, 7], who used Kato's theory of smooth operators and Putnam's commutator method. Lavine's results have been extended recently by Robinson [8] who considered positive decreasing potentials with an arbitrary singularity at the origin (see also Semenov [9]).

It is the aim of the present note to prove (KC) for potentials of the form \( W + V \), where \( W \) is
more or less a standard potential, i.e. for \( l = 3 \) \( W(x) \in \mathcal{A} \), the Rollnik class [4] and for \( l > 4 \) \( W(x) \in (L^2 \cap L^p)(\mathbb{R}^l), p > l/2 \) and \( V \) is a positive strongly singular potential: \( V(x) \in L_{loc}^{2\alpha}(\mathbb{R}^l) \), where \( S \) is an arbitrary closed bounded set of Lebesgue measure zero.

The method to prove our result is very simple and consists in the following. First we prove that for any \( \alpha > 0 \) \( e^{-tH_0} - e^{-t(H_0 + V)} \) belongs to trace class. To do this we apply some results about generalized convergence of operators \( H_0 + V_n \), where \( \{V_n\} \) is some suitable approximation of \( V \) and use one comparing feature in the theory of parabolic equations.

Then we establish (KC) for \( \Omega_\alpha(H_0 + V, H_0) \) appealing to the invariance principle:
\[
\Omega_\alpha(e^{-(H_0 + V)}, e^{-H_0}) = \Omega_\alpha(H_0 + V, H_0)
\]
and the theorem of Birman [5].

The next step consists in proving the existence and (KC) for \( \chi^2_\alpha(H_0 + W + V, H_0 + V) \). This requires some quadratic estimates of the type \( ||W(H_0 + V + \lambda)^{-1}|| \leq ||W(H_0 + \lambda)^{-1}|| \) and familiar trace arguments. The result now will follow from the chain rule for wave operators.

Our main result is

**THEOREM.** Let \( \mathcal{H} = L^2(\mathbb{R}^l), l \geq 3 \). Let \( q = W + V \). Suppose that

1. \( 0 \leq V(x) \in L_{loc}^{2\alpha}(\mathbb{R}^l) \), supp \( V(x) \subset K(R) \), where \( S \) is a closed set of measure zero and \( K(R) = \{x \in \mathbb{R}^l; |x| \leq R \} \) for some fixed \( R > 0 \).

2. For \( l = 3 \) \( W(x) \in \mathcal{A} \), the Rollnik class and for \( l > 4 \) \( W(x) \in (L^2 \cap L^p)(\mathbb{R}^l) \) for some \( p > l/2 \). Let \( H = H_0 + q \) be the form sum. Then \( \chi^2(H_0 + V, H_0 + V) \) exist and are (KC).

**Remark.** The condition ‘supp \( V(x) \subset K(R) \)’ in the theorem may be replaced, for instance, by the condition \( V(x) = O(|x|^{-2+\epsilon}) \) for \( l = 3 \) and \( V(x) = O(|x|^{-l/2}) \) for \( l > 4 \) if \( |x| \to \infty \) in which case \( V \) can be decomposed as the sum \( V_c + V_w \) where \( V_c \) has a compact support and \( V_w \) satisfies Condition 2).

2. **PROOF OF THE THEOREM**

Let \( V_n \) be the corresponding truncated operators of \( V \), i.e. \( V_n = V \) if \( V \leq n \), \( V_n = n \) if \( V > n \).

**LEMMA 1.** Let \( \mathcal{H} = L^2(\mathbb{R}^l), l \geq 1 \). Let \( 0 \leq V(x) \in L_{loc}^{1\alpha}(\mathbb{R}^l) \), where \( S \) is an arbitrary closed set of measure zero. Let \( H_n = H_0 + V_n \). Then for every \( \alpha > 0 \), \( s \lim_{n \to \infty} e^{-tH_n} = e^{-tH_0} \), where \( H_\alpha = H_0 + V \) denotes the form sum.

**Proof.** A detailed proof of the lemma can be found in [10].

Let \( V(0)(x) : \mathbb{R}^l \to \mathbb{R}^1 \) be a nonnegative function which vanishes outside of \( K(2R) \) and is infinity on \( K(R) \). The function \( V(0)(x) \) will be called a comparable potential if and only if \( V_n(0)(x) \) is locally Holder continuous function on \( \mathbb{R}^l \) for each \( n = 0, 1, 2, ... \).

Let \( V(0) \) be a comparable potential. Let \( H(0)_n = H_0 + V_n(0) \). Then by an argument based on monotony of the sequence \( \{H(0)_n\} \) it is easy to see that for any \( \alpha > 0 \), \( s \lim_{n \to \infty} e^{-tH(0)_n} \) exist and uniquely defines the semigroup \( P^t \) of bounded self-adjoint operators acting in \( \mathcal{H} \).

**LEMMA 2.** Let \( \mathcal{H} = L^2(\mathbb{R}^l), l \geq 1 \). Let \( V(0) \) be a comparable potential and let \( K(0)(x, y, t) \) and \( K(0)(x, y, t) \) are the integral kernels of \( e^{-tH_0} \) and \( P^t \) respectively. Then for each \( x \in \mathbb{R}^l \) the following estimate is valid