QUANTUM MECHANICS AS A DEFORMATION OF CLASSICAL MECHANICS*

F. BAYEN
Département de Mathématiques, Université de Paris 6, 75230 Paris Cedex 05, France

M. FLATO** and C. FRONSDAL
University of California, Los Angeles, Calif. 90024, U.S.A.

and

A. LICHNEROWICZ and D. STERNHEIMER
Collège de France, 75231 Paris Cedex 05, France

ABSTRACT. Mathematical properties of deformations of the Poisson Lie algebra and of the associative algebra of functions on a symplectic manifold are given. The suggestion to develop quantum mechanics in terms of these deformations is confronted with the mathematical structure of the latter. As examples, spectral properties of the harmonic oscillator and of the hydrogen atom are derived within the new formulation. Further mathematical generalizations and physical applications are proposed.

1. INTRODUCTION

The probabilistic interpretation of quantum mechanics has led to attempts to interpret it as a statistical theory over phase space. Wigner [1] gave an expression for a phase space distribution function, related to Weyl's quantization procedure [2]. (See also [3], [4] and further references given there.) An extensive study was made by Moyal [5], who introduced the 'sine Poisson' bracket, usually called Moyal bracket, for functions on phase space.

Some of us [6] gave a complete study of the 1-differentiable deformations of the Poisson bracket on a symplectic manifold and developed some physical applications. Such deformations are trivial in the flat case (manifold $\mathbb{R}^{2l}$ with the ordinary symplectic structure). Later, Vey [7] exhibited in the flat case a nontrivial deformation by differentiable cochains of unbounded order and demonstrated the existence of similar deformations, globally defined on general symplectic manifolds with vanishing 3-cohomology. In the flat case the Vey bracket coincides with the Moyal bracket. A more detailed study of this deformation, including mathematical properties and physical applications, were sketched in [8].

---

*Work supported in part by the National Science Foundation.
**Permanent address: Physique Mathématique; Collège de France, and Université de Dijon, 21000 Dijon, France.

Letters in Mathematical Physics 1 (1977) 521-530.

An important feature of some deformed Poisson brackets is that they can be derived from an associative algebra, itself a deformation [9] of the ordinary algebra of functions on phase space. One calls twisted product or *-product the composition law of this deformed algebra.

The problem of quantization has been treated geometrically by Souriau [10], Kostant [11] and others. *We propose to treat quantum theories as deformations of classical theories.* Quantization manifests itself as a deformation of the structure of the algebra of classical observables, rather than a change in the nature of the observables. It turns out that spectral properties and other concepts usually associated with representation theory appear naturally in this formalism.

2. SOME MATHEMATICAL PROPERTIES OF THE MOYAL-VEY DEFORMATIONS

(a) Let W be a differentiable symplectic manifold of dimension $2l$ with symplectic 2-form F, N the differentiable functions on W, μ the standard isomorphism from the tangent bundle TW to the cotangent bundle T^*W defined by $\mu(X) = -i(X)F$ (i: interior product) and extended to tensors, so that $\Lambda = \mu^{-1}(F)$ defines the Poisson bracket by

$$\{u, v\} = i(\Lambda)\,(\mathrm{d}u \wedge \mathrm{d}v) \equiv P(u, v), \qquad \text{for} \qquad u, v \in N$$

cf.e.g. [8] and [6]). On W there exist symplectic connections Γ, namely linear connections without torsion such that $\nabla\Lambda = 0$, where ∇ is the operator of covariant differentiation defined by Γ. We may then define the order r (in each argument) bidifferential operators P^r by their expressions on a local chart $U \subset W (i, j = 1, \ldots, 2l)$:

$$P^r(u, v)|_U = \Lambda^{i_1 j_1} \ldots \Lambda^{i_r j_r} \nabla_{i_1 \ldots i_r} u \, \nabla_{j_1 \ldots j_r} v. \tag{1}$$

When $W = \mathbb{R}^{2l}$ with the usual symplectic structure and connection, P^r is the rth power of P.

In the flat case (i.e. when there exists a Γ without curvature, and if we do make this choice) one obtains a formal deformation (in the sense of [9]), of the associative algebra defined on N by the usual product, with the twisted product $*_\lambda$ that can be written symbolically (P^r being defined by (1)):

$$u *_\lambda v = \exp(i\lambda)(u, v) \equiv uv + \sum_{r=1}^{\infty} ((i\lambda)^r/r!)\, P^r(u, v). \tag{2}$$

One is thus led to consider the Lie bracket, formal deformation of P:

$$M(u, v) = (2i\lambda)^{-1}\,(u *_\lambda v - v *_\lambda u) = \lambda^{-1} \sin(\lambda P)\,(u, v), \tag{3}$$

which is the Moyal bracket [5] for the usual $W = \mathbb{R}^{2l}$ nad $\lambda = \hbar/2$.

In the general case, with the technical condition $H^3(W; \mathbb{R}) = 0$ (de Rham cohomology), Vey [7] showed that there exists a deformation of the Lie algebra N (with bracket P) given by ($\nu = -\lambda^2$):