DEFORMATIONS OF REPRESENTATIONS

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ABSTRACT. A connection between deformation of Lie group representations and deformations of associated Lie algebra representations is established. Applications are given to the theory of analytic continuation of K-finite quasi-simple representations of semi-simple Lie groups. A construction process of all TCI representations of \( SL(2, R) \) is obtained.

INTRODUCTION

Given a representation \( U \) of a Lie group \( G \) in a Banach space \( A \), let us define a deformation of \( U \) by a one parameter family \( U_{g, \lambda} \in G, \lambda \in \mathbb{C}, |\lambda| < R \neq 0 \) of representations of \( G \) (in \( A \)) which is the sum of a power series \( U_{g, \lambda} = U_g + \sum_{n>0} \frac{\lambda^n}{n!} T^n_{g, \lambda} \in \mathcal{L}(A) \), with radius of convergence \( R \) in \( \mathbb{C} \) (see Def. (1.1)).

In the first section, we establish some general features of deformations. A notion of equivalent deformations is introduced (Def. (1.4)). We mainly use this equivalence to replace the initial deformation by an equivalent one, which satisfies differentiability conditions (Prop. (1.6)); consequently, we establish a connection between deformations of representations of the group \( G \) and deformation of associated representations of its Lie algebra \( \mathfrak{g}_0 \) (Prop. (1.7): the differential of the (regularized equivalent) deformation is obtained on \( C^\infty \) vectors by differentiation, term by term, of this power series, with respect to the group variable. As a corollary ((1.8)), we characterize the differentiable vectors of a deformation, in term of the differentiable vectors of the initial representation \( U \). A converse of this process (in the simply connected case) is stated by Prop. (1.10), which gives conditions in order to 'integrate' a deformation of a Lie algebra representation to a deformation of the associated group representation.

In the second section, (keeping in mind the useful theory of analytic continuation of the principal series (e.g.: [10])) we specialize to the semi-simple case. We first show that (up to equivalence) deformations are always trivial when restricted to a maximal compact subgroup \( K \) of \( G \) (Prop. (2.1)). Consequently deformations of \( K \)-finite representation remain \( K \)-finite (Corollary (2.1)), with a 'constant' restriction to \( K \) (see also corollary (2.2) for analytic continuations). Propositions (2.4) and (2.5) give some useful conditions under which a strengthened form of the integrability Theorem (1.10) is valid: this is stated in Proposition (2.8) in the case of \( K \)-finite quasi-simple representations. As a corollary ((2.9)), we establish a construction process of analytic continuation of a group representation: roughly speaking, analytic continuation of the parameters, in a Lie algebra repre-

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sentation, leads to analytic continuation of the associated group representation. Let us mention that this result is quite obvious to apply in examples. Finally, we study eventual continuations of a deformation to points which belong to the limit disc of convergence (Proposition (2.10)).

If we consider the particular case of \( G = SL(2, \mathbb{R}) \), starting from suitably normalized representation of the principal series, we can construct two families \( U^\lambda \) and \( U^q_{1/2}, q \in \mathbb{C} \), of representations using (2.8), (2.9), (2.10). \( q \) is the value of the Casimir element. \( U^\lambda \) is an analytic continuation of the initial representation in the domain \( \Omega = \{ q \in \mathbb{C} | q \notin \mathbb{R}^- \} \), and the values \( U^\lambda, q \in \mathbb{R}^- \), are obtained when extending by continuity to the frontier of the domain. Similar results hold for \( U^q_{1/2} \) with \( \Omega = \{ q \in \mathbb{C} | q \notin \{ x \in \mathbb{R} | x \in 1/4 \} \}. \) This construction provides all unitary irreducible representations of \( G \): (normalized principal series are realized by \( U^\lambda \) and \( U^q_{1/2} \) when \( q > 1/4 \) and complementary series by \( U^\lambda \) when \( 0 < q \leq 1/4 \); discrete series appears in the splitting of \( U^\lambda \) and \( U^q_{1/2} \) at the singular points \( q = -n(n + 1) \), resp.: \( q = -(1/2 + n)(1/2 + n + 1), n \in \mathbb{Z} \).

Moreover, this process provides a Hilbert space realization of every T.C.I. representation of \( G \). In particular, finite dimensional irreducible representations appear as a summand of the splitting at the mentioned singular points.

Finally, let us mention the following problem, which is not developed in the present article, though most of the proofs have an implicit cohomological nature. For finite dimensional representations, deformation theory was intensively studied (e.g.: [7]), and meaningful results obtained, such as rigidity when some cohomology group \( H^1 \) vanishes. Is there a corresponding result in the infinite dimensional case? Let us remark that, in a formal way, there seems to be no critical obstruction to generalize (to the infinite dimensional case), provided that one chooses an adapted \( H^1 \) (e.g.: [4] for the formal part, and [8] for the adapted \( H^1 \)).

Let \( G \) be a connected Lie group, \( \mathfrak{g} \) its (real) Lie algebra, \( \mathfrak{g} \) the complexification of \( \mathfrak{g} \), and \( \mathfrak{g}(\mathfrak{g}) \) the universal enveloping algebra of \( \mathfrak{g} \). If \( (U, A) \) is a (continuous) linear representation of \( G \) in a complex Banach space \( A \), we denote by \( A^\infty \) the space of differentiable vectors of \( U \) endowed with its usual Fréchet space topology. \( dU \) stands for the corresponding representation of \( \mathfrak{g} \) (or \( \mathfrak{g} \)) on \( A^\infty \). We denote by \( \mathcal{L}(A) \) the Banach space of continuous linear mappings from \( A \) into \( A \), and by \( \mathcal{L}(A^\infty) \) the space of continuous linear mappings from (the Fréchet space) \( A^\infty \) into \( A^\infty \).

## 1. DEFORMATIONS

Let \( \lambda \in \mathbb{C}, R \in \mathbb{R} \), we denote by \( B(\lambda, R) \) the open ball (in \( \mathbb{C} \)) of center \( \lambda \) and radius \( R \).

**DEFINITION (1.1).** By a deformation of \( (U, A) \), we mean a mapping \( \lambda \mapsto U^\lambda_g, g \in G \) from \( B(0, R), R \neq 0, \) into \( \mathcal{L}(A) \) such that:

1. For every \( \lambda \in B(0, R) \), \( U^\lambda \) is a linear representation of \( G \) in \( A \)
2. There exists a power series \( \sum_{n > 0} \frac{\lambda^n}{n!} T^n_g, g \in G \), with radius of convergence \( R \) in \( \mathcal{L}(A) \) such that
\[
U^\lambda_g = \sum_{n > 0} \frac{\lambda^n}{n!} T^n_g, \lambda \in B(0, R) \) (obviously \( T^0 = U \)).

**Remark.** The coefficients \( T^n \) satisfy: \( T^n_{gg'} = \sum_{i=0}^n C_i^n T^i_g T^{n-i}_{g'} \), \( n \in \mathbb{N}, g, g' \in G \).

**PROPOSITION (1.2).** The mapping \( (g, \varphi) \mapsto T^n_g(\varphi) \) is continuous from \( G \times A \) into \( A \).