Classification of all Poisson–Lie Structures on an Infinite-Dimensional Jet Group

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Abstract. A local classification of all Poisson–Lie structures on an infinite-dimensional group $G_{\infty}$ of formal power series is given. All Lie bialgebra structures on the Lie algebra $g_{\infty}$ of $G_{\infty}$ are also classified.


Key words: Poisson–Lie group, Lie bialgebra, Yang–Baxter equation.

Let $G_{\infty}$ be the group of formal power series in one variable

$$\left\{ x(u) = \sum_{i=1}^{\infty} x_i u^i \bigg| x_1 \neq 0 \right\},$$

with a group multiplication $G_{\infty} \times G_{\infty} \to G_{\infty}$ being the substitution

$$(xy)(u) := x(y(u)) \quad \text{or} \quad u \mapsto x(u), \quad (1)$$

with an identity $e$ being the identity map $u \mapsto u$. The group $G_{\infty}$ is the group of formal diffeomorphisms of $\mathbb{R}^1$ which leave the origin fixed. It is a projective limit $G_{\infty} = \lim_{n} G_n$, where $G_n$ are the finite-dimensional Lie groups of $n$-jets of the line at the origin. The multiplication in $G_n$ is again defined by the substitution (1):

$$(\mathcal{X}_n \mathcal{Y}_n)(u) := \mathcal{X}_n(\mathcal{Y}_n(u)) \mod u^{n+1}, \quad \text{where} \quad \mathcal{X}_n(u) \text{ and } \mathcal{Y}_n(u) \text{ are polynomials in } u \text{ of degree } n.$$ 

We define the space of smooth functions $C^{\infty}(G_{\infty})$ to be the inductive limit $C^{\infty}(G_{\infty}) = \lim_{n} C^{\infty}(G_n)$ of the spaces of smooth functions on the finite-dimensional groups $G_n$.

Following [1], we consider a multiplicative Poisson (Poisson–Lie) structure on $G_{\infty}$ to be the bilinear skew-symmetric map \{ , \} $C^{\infty}(G_{\infty}) \times C^{\infty}(G_{\infty}) \to C^{\infty}(G_{\infty})$ defined by

$$\{f,g\} = \omega_{ij} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}, \quad (2)$$
for any \( f, g \in C^\infty(G_\infty) \), such that the multiplication map \( G_\infty \times G_\infty \to G_\infty \) is a Poisson map. Here \( \omega_{ij} \in C^\infty(G_\infty) \), for any \( i, j \in \mathbb{N} \), and a summation is assumed over repeated indices. The Poisson structure on \( G_\infty \times G_\infty \) is taken to be the product Poisson structure. Note that the sum in (2) is finite, since by definition, \( f \) and \( g \) are functions of a finite number of variables. Then the Jacobi identity for \( \{ \, , \} \) implies that the \( \omega_{ij} \)'s satisfy

\[
\frac{\partial \omega_{kl}}{\partial x_i} + \omega_{ik} \frac{\partial \omega_{lj}}{\partial x_i} + \omega_{il} \frac{\partial \omega_{jk}}{\partial x_i} = 0,
\]

for any \( j, k, l \in \mathbb{N} \). The multiplicativity of the Poisson brackets (2) (\( \{ \, , \} \) being a 1-cocycle) means that \( \omega_{ij} \)'s must satisfy the following infinite system of functional equations

\[
\omega_{ij}(xy) = \omega_{kl}(x) \frac{\partial z_i}{\partial x_k} \frac{\partial z_j}{\partial x_l} + \omega_{kl}(y) \frac{\partial z_i}{\partial y_k} \frac{\partial z_j}{\partial y_l}, \quad i, j \in \mathbb{N},
\]

where \( z = xy \). Note again that the sums in the right-hand side of (3b) are finite.

This is immediately seen from the explicit formulae

\[
z_k = \sum_{i=1}^{k} x_i \sum_{\sum_{\alpha \in A_0} = k} y_{j_1} \cdots y_{j_n}, \quad k \geq 1,
\]

for the coordinates of \( z \). From (3b), it also follows that \( \omega_{ij}(e) = 0 \).

Do such structures exist on \( G_\infty \)? It is by no means obvious that such structures do exist. For example:

(i) Let us consider the three-dimensional factor group \( G_3 = G_\infty (\bmod \) \( w^n \), for \( n \geq 4 \). Then there exists a Poisson–Lie structure on \( G_3 \) described by

\[
\{ x_1, x_2 \} = x_1 x_2, \quad \{ x_1, x_3 \} = 4x_2^2 - 2x_1 x_3,
\]

\[
\{ x_2, x_3 \} = 6x_3^2 - 5x_2 x_3,
\]

where \( x_1, x_2, x_3 \) are the coordinate functions on the group \( G_3 \). However, this Poisson–Lie structure cannot be extended to a Poisson–Lie structure on \( G_\infty \).

(ii) We conjecture that there are no nontrivial Poisson–Lie structures on the group of diffeomorphisms of \( S^1 \).

Also a second question arises: If such structures exist, could they be classified? The answer to the first question is given by the following theorem.

**THEOREM 1.** For every natural number \( d \in \mathbb{N} \), and every sequence \( M_d = (\mu_n)_{n=1}^\infty \), such that \( \mu_n = 0 \) for \( 1 \leq n \leq d \) and \( \mu_{d+1} \neq 0 \), one has the following infinite-parameter family of Poisson–Lie structures on \( G_\infty \),

\[
\omega_{ij}(x) = \sum_{p=1}^{i} \sum_{q=1}^{j} px_p q x_q \lambda_{i-p+1,j-q+1} - \lambda_{i-j}\lambda_{i-j-1} - \lambda_{i-j+1},
\]

\( i, j \in \mathbb{N} \).