ABSTRACT. We introduce an extension of the concept of variational derivative which allows us to derive an integral identity in the variational formalism. It is applied to characterize the kernel and range of the variational derivative and the divergence operator.

1. INTRODUCTION

This note deals with several basic aspects of the variational formalism. The study of the kernel and range of the variational derivative is based on a natural extension of the concept of variational derivative which have very useful properties. In this way, we find (Lemma 1) that an identity implicit in the work of Ibragimov [1] becomes simple when expressed in terms of the new variational derivatives. This allows us to derive a striking integral identity (Lemma 2) from which follows at once (Theorems 1, 2 and 3) the characterization of the kernel and range of the variational derivative and of the total divergence operator.

The results of our work are derived in a general context of sufficiently regular functions $F[x, u]$ depending upon $n$ independent variables $x_i (i = 1, ..., n)$ and derivatives of arbitrary order of $m$ dependent variables $u^r (r = 1, ..., m)$. In particular we state the equivalence between Ker $\frac{\delta}{\delta u}$ and Ran $\vec{D}$ which plays a central role in the analysis of the conservation laws associated with partial differential equations. We notice that this equivalence is deduced from the Gel'fand-Dikii symbolic calculus [2] only for polynomial functions $P[u]$ with $n = 1$ and $m = 1$. On the other hand, our characterization of Ran $\frac{\delta}{\delta u}$ provides a simple method to test when a system of partial differential equations derives from a variational principle. It is a new criterion which differs from the usual one based on the symmetry of the Frechet derivative [3].

2. VARIATIONAL DERIVATIVES

Let us consider a variational formalism with $n$ independent variables $x = (x_1, ..., x_n)$ and $m$
dependent variables \( u = (u^1, \ldots, u^m) \). For each ordered set \( \alpha = (\alpha_1, \ldots, \alpha_n) \) of \( n \) non-negative integers, we denote
\[
|\alpha| = \sum_{i=1}^{n} \alpha_i, \quad u^\alpha = \frac{\partial^{\alpha_1} u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}. 
\]

Given \( \alpha = (\alpha_1, \ldots, \alpha_n) \) and \( \beta = (\beta_1, \ldots, \beta_n) \), we shall write \( \beta \preceq \alpha \) if \( \beta_i \leq \alpha_i \) for all \( i = 1, \ldots, n \). In this case we define:
\[
\binom{\alpha}{\beta} = \prod_{i=1}^{n} \left( \frac{\alpha_i}{\beta_i} \right) = \prod_{i=1}^{n} \frac{\alpha_i!}{\beta_i!(\alpha_i - \beta_i)!}. 
\]

We shall consider functions \( F = F[x, u] \) depending upon an arbitrary finite subset of the variables \((x_i, u^1, \ldots, u^m)\). It will be tacitly assumed that these functions are sufficiently regular to render meaningful the differentiation operations involved in what follows. In particular, they are in the domain of total derivative operators of the form:
\[
D^\alpha = \prod_{i=1}^{n} \left( \frac{\partial}{\partial x_i} + \sum_\beta \frac{\partial u^\beta}{\partial x_i} \frac{\partial}{\partial u^\beta} \right)^{\alpha_i}. 
\]

The variational derivative with respect to the variable \( u^\alpha \) will be here defined as
\[
\frac{\delta F}{\delta u^\alpha} = \sum_\beta (-1)^{|\beta|} \binom{\alpha + \beta}{\beta} D^\beta \frac{\partial F}{\partial u^{\alpha + \beta}}. 
\]

In particular \( \delta / \delta u^\alpha \) coincides with the usual variational derivative:
\[
\frac{\delta F}{\delta u^\alpha} = \sum_\beta (-1)^{|\beta|} D^\beta \frac{\partial F}{\partial u^\beta}. 
\]

Our choice (4), which clearly departs from common practice, is motivated by two facts: first, its translation to Gelfand-Dikii algebraic calculus [2] yields a simple representative [4] which naturally generalizes that of (5); and second, it renders neater the Ibragimov identity [1] (see (6) below).

The interest of the operators (4) is made evident by the following properties:

**Lemma 1.** Given \( \eta^r[x, u] \) \((r = 1, \ldots, m)\) and \( F[x, u] \), the following identity holds:
\[
\sum_\alpha D^\alpha \eta^r \frac{\partial F}{\partial u^\alpha} = \sum_\alpha D^\alpha \left( \eta^r \frac{\partial F}{\partial u^\alpha} \right). 
\]

*Proof.* Let us note that, with our notation, Leibnitz rule writes:

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