Abstract. The concept of vector spherical harmonics is generalized for symmetric and traceless Cartesian tensor fields of arbitrary rank. Differential relations of these functions are derived as generalizations of the gradient formula for scalar, and the divergence and curl formulas for vector spherical harmonics.

The irreducible representations of the rotation group by vector fields, known as vector spherical harmonics [1, 2], form an appropriate tool for angular-momentum classification of space-dependent vector quantities, such as electromagnetic fields [3] or velocity fields in fluid mechanics [4]. Their extension to more general tensor fields is straightforward, as long as one contents oneself with the usual coupling of orbital angular momentum to an internal angular momentum represented by a standard set of spin eigenfunctions.

An elaborate instrument for analyzing fields of higher spins, which does not imply a coupled angular momentum, has been developed as spin-weighted spherical harmonics [5, 6]. On the other hand, to our knowledge, no general explicit calculus exists for integer-number representations of a definite total angular momentum, where the internal rotational behaviour is characterized by Cartesian tensors. Special tensor spherical harmonic functions of rank 2 have been constructed in the framework of gravitation theory [7]. A more general formalism might be of use, for instance, in kinetic theories, where the Cartesian representation of higher moments of the local momentum distribution functions, generalizations of the velocity-field, seems to be more convenient than the spherical one (e.g., see Reference [8]).

In this letter we propose an extension of the concept of vector spherical harmonics to Cartesian tensor fields of arbitrary rank, which, of necessity, must be symmetrical and traceless. In particular, we derive elementary relations, which express the coupling of the coordinate and gradient vector to tensor spherical harmonics by these same functions, thus allowing application of the calculus to expansions in differential equations, such as kinetic equations for spherically-symmetric situations.

Starting from Cartesian unit-vectors \( e_\alpha \), with \( \alpha = 1, 2, 3 \) for \( x, y, \) and \( z \) direction, and in the \( s \)-fold tensor-product space generated by this set, we define linear combinations of the tensor basis that form a \( 2s + 1 \)-dimensional irreducible representation of the rotation group:

\[
C^{s m}_p = C^{s m}_m e^{(1)}_\alpha \otimes \ldots \otimes e^{(s)}_\alpha \quad m_s = -s, \ldots, s; \quad s = 0, 1, \ldots .
\]  

(1)

* A summation convention is used.
Due to coupling to the maximal spin $s$, the Cartesian components $C^{\alpha s}_{\alpha s}$ of these tensors are symmetric and traceless functions of their indices. In general of the meaning of generalized coupling coefficients, for $s = 1$ they just transform a vector from Cartesian to spherical axes,

$$C^{1+1}_{\alpha} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ -i \\ 0 \end{pmatrix}, \quad C^{1,0}_{\alpha} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad C^{1,-1}_{\alpha} = \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}.$$  \tag{2}

In a second coupling, we linearly combine spin tensors (1) of a definite $s$ with spherical harmonic functions $Y^{lm}_{i}(\Omega)$ of a definite value of $l$, to a standard set of $(j, m)$ eigentensors of the total angular momentum,

$$Y^{(ls)lm}(\Omega) = \sum_{m_{l}, m_{s}} \binom{l}{m_{l}} \binom{s}{m_{s}} \binom{j}{m} Y^{ilm}_{i}(\Omega) C^{sms}_{m_{l}, m_{s}}$$ \tag{3}

the coefficients of this coupling being the standard vector-coupling coefficients. The Cartesian components of the tensor (3) may be called the (Cartesian) tensor spherical harmonics

$$Y^{(ls)lm}(\Omega) = \sum_{\alpha_{1}, \ldots, \alpha_{s}} \binom{l}{m_{l}} \binom{s}{m_{s}} \binom{j}{m} Y^{ilm}_{i}(\Omega) C^{sms}_{\alpha_{1}, \ldots, \alpha_{s}}; \tag{4 a-c}$$

In (4c) a compact notation of angular momentum coupling is defined to be used in the following.

Like their constituents, $C^{sms}_{\alpha_{1}, \ldots, \alpha_{s}}$, the tensor spherical harmonics are symmetric and traceless tensors (we use this expression for the ensemble of components henceforth). For $s = 0$ and $s = 1$, they reduce to the standard scalar and vector spherical harmonics. As in these special cases, they in general split into two groups of normal parity, $(-1)^{s+l}$, and anomalous parity, $(-1)^{l+s+1}$, corresponding to the odd and even values of $l-j$. The tensor spherical harmonics of a given value of $s$ are orthonormal in the meaning of the following scalar product,

$$\int d\Omega Y^{(ls)lm}_{\alpha_{1}, \ldots, \alpha_{s}}(\Omega) Y^{(ls')m'}_{\alpha_{1}, \ldots, \alpha_{s}}(\Omega) = \delta_{l,l'}\delta_{m,m'}. \tag{5}$$

This is proved successively starting from the orthogonality of the $Y^{lm}$, and utilizing properties of the vector coupling coefficients as well as the coupling relation, true by definition of the $C^{s}$,

$$C^{s+1, m_{s+1}}_{\alpha_{1}, \ldots, \alpha_{s+1}} = [C^{1}_{\alpha_{s+1}}, C^{s}_{\alpha_{1}, \ldots, \alpha_{s}}]^{s+1, m_{s+1}}.$$ \tag{6}

To investigate the effect of the coordinate and the gradient vector on the tensor spherical