TRANSITION MATRICES OF SELF-DUAL SOLUTIONS TO SU(N) GAUGE THEORY

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ABSTRACT. We extract self-dual potentials from SU(N) transition matrices of the Atiyah–Ward type. The simplest nontrivial example is studied in detail and we find a topologically nontrivial, regular, self-dual solution. Different gauges can be found for which the corresponding gauge potentials are respectively time-independent or real.

Since 't Hooft and Polyakov [1] found the first Yang–Mills–Higgs monopole, a large amount of work has gone into the search for multimonompole solutions. However, it was only recently that multimonompole solutions were found. This important progress is due to Taubes [2], Ward [3], Prasad and Rossi [4] and Forgacs et al. [5]. Generalizing Ward's work [6] on two monopole solutions, Corrigan and Goddard [7] were able to find static SU(2) monopole solutions with the maximal number of degrees of freedom for self-dual solutions. These solutions may well be the general self-dual SU(2) monopole solutions for arbitrary magnetic charge, provided that the proof of regularity can be completed.

Now that all self-dual monopole solutions to SU(2) gauge theory seem to have been found, it is natural to investigate the SU(N) monopole solutions. In order to find these, a direct approach would be to generalize the SU(2) technique based on the work by Atiyah and Ward [8] which Corrigan et al. [9] elaborated and Corrigan and Goddard successfully applied in Reference [7]. It is exactly this that is developed in the following.

We start out with an SU(N) transition matrix which is a generalization of the one proposed for SU(2) by Atiyah and Ward, namely

\[
g = \begin{bmatrix}
\zeta^l_1 & \rho_{12} & \cdots & \rho_{1N} \\
0 & \zeta^l_2 & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & \zeta^l_N
\end{bmatrix} \in SL(N, C),
\]

with \( l_n \in \mathbb{Z}, l_1 = -\sum_{n=2}^{N} l_n \) and Laurent series \( \rho_{mn}(x_{ab}, \zeta) \) in \( \zeta \). The \( \rho_{mn} \) are required to satisfy the condition

\[
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\[ D_{\alpha} \rho_{mn} = \left( \frac{\partial}{\partial x_{a_1}} - \xi \frac{\partial}{\partial x_{a_2}} \right) \rho_{mn} = 0 \]  

and \( x_{ad} (\alpha, \beta = 1, 2) \) are the matrix elements of \( x = x_0 - i x \cdot a, x_{\mu} (\mu = 0, 1, 2, 3) \) are complexified coordinates and \( a_i \) Pauli matrices.

If we choose \( N = 3 \) with \((l_3 = 0, l_2 = -l, \rho_{13} = \rho_{23} = 0, \rho_{12} = \rho), (l_2 = -l_3 = l, \rho_{12} = \rho_{13} = 0, \rho_{23} = \rho) \) or \((l_2 = 0, l_3 = -l, \rho_{12} = \rho_{23} = 0)\), the ansatz (1) reduces to the \( I, U \) and \( V \)-spin embedding of \( SU(2) \) and \( SU(3) \), respectively. The maximal embedding, however, is not of this form. In fact, with the help of the formula

\[ R_{ij} = \frac{1}{2} \text{tr} (a_i a^{-1} a_j) \]

where \( a \) and \( R \) mean \( a = \exp (a \cdot \sigma) \) and \( R = \exp (a_1 \lambda_1 - a_2 \lambda_5 + a_3 \lambda_2) \), respectively, with Gell-Mann matrices \( \lambda_i \), we can calculate the transition matrix for the maximal embedding. The resulting matrix is

\[ g = \begin{pmatrix}
\frac{1}{2} (\rho^2 + t^2 + y^2) & \frac{1}{2i} (\rho^2 + t^2 - y^2) & y - i t \\
\frac{1}{2i} (\rho^2 - t^2 + y^2) & \frac{1}{2} (\rho^2 + t^2 + y^2) & i t - i y \\
-\frac{1}{2} t & -i \frac{1}{2} y & 1
\end{pmatrix} \]

and we do not know whether it is gauge equivalent to a special case of (1). Even if it is, it can look rather complicated in this gauge. If we restrict our attention to the ansatz (1), it is therefore only for the reason that we can extract self-dual gauge potentials \( A_\mu \in \text{sl}(N, \mathbb{C}) \) from it.

In order to do this we generalize the Corrigan et al. [9] technique: We split

\[ g(x, \xi) = h(x, \xi) k^{-1}(x, \xi), \]

where \( h \in \text{SL}(N, \mathbb{C}) \) is a Taylor series in \( \xi^{-1} \) and \( k \in \text{SL}(N, \mathbb{C}) \) a Taylor series in \( \xi \). Equation (5) then yields the recursion formulas

\[ k_{mn} = \sum_{r=0}^{\infty} h_{mn}^{r+lm} \xi^r - \sum_{p=m+1}^{N} \sum_{r=0}^{\infty} \sum_{s=-r}^{\infty} k_{pn}^{r} \rho_{mp}^{s} \xi^{r+s-lm}, \]

\[ h_{mn} = \sum_{r=l_m}^{0} h_{mn}^{r} \xi^r + \sum_{p=m+1}^{N} \sum_{r=0}^{\infty} \sum_{s=-r}^{\infty} k_{pn}^{r} h_{mp}^{s} \xi^{r+s}, \]

\((m = N, ..., 1)\) and the constraints

\[ \sum_{r=0}^{\infty} \sum_{p=m+1}^{N} k_{pn}^{r} \rho_{mp}^{r+s} = 0 \]