A LOWER BOUND FOR THE TRANSITION PROBABILITY DENSITY OF A DIFFUSION PROCESS

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ABSTRACT. We give a lower bound for the transition probability density of a diffusion process in terms of the transition probability density of an auxiliary process.

1. INTRODUCTION

Let \((\Omega, F, \mu)\) be a probability space and \(\nu\) another probability measure on \((\Omega, F)\), which is equivalent to \(\mu\). The generalized entropy (information gain) of \(\mu\) with respect to \(\nu\) is defined by

\[
S(\mu \| \nu) = - \int_{\Omega} d\nu(\omega) \frac{d\mu}{d\nu}(\omega) \ln \frac{d\mu}{d\nu}(\omega).
\]

In [1] we have applied this concept to measures induced by diffusion processes as follows. Let \(X_s, Y_s, s \in [0, t]\) one-dimensional diffusion processes with state space \(K \subset \mathbb{R}\), which are defined by the stochastic differential equations of an Itô type [2]

\[
\begin{align*}
\text{d}X_s &= f(X_s) \text{d}s + G(X_s) \text{d}W_s, \quad X_0 = x_i \in K \\
\text{d}Y_s &= k(Y_s) \text{d}s + G(Y_s) \text{d}W_s, \quad Y_0 = y_i \in K.
\end{align*}
\]  

(2)

Here \(W_s, s \in [0, t]\), denotes the standard Wiener process and we assume that \(t < \min(t_X, t_Y)\), where \(t_X, t_Y\) denotes the explosion time of the process \(X_s\) and \(Y_s\), respectively.

We denote, by \(\mu_X, \mu_Y\), the measures induced on the measurable space \((C(x_i), C(x_i) \cap \mathcal{B}(C([0, t]))))\) by \(X_s\) and \(Y_s\) respectively. Here \(C(x_i)\) is a subset of the space of continuous functions \(C([0, t])\) on \([0, t]\)

\[
C(x_i) = \{ \xi \in C([0, t]) \mid \xi(0) = x_i \}
\]

(3)

and \(\mathcal{B}(C([0, t]))\) denotes the Borel field generated by the open subsets of \(C([0, t])\), with respect to the uniform norm.

Due to the fact that the diffusion terms, as well as the initial conditions of the processes \(X_s\) and
$Y_z$ are identical, $\mu_X$ is equivalent to $\mu_Y$ and the Radon–Nikodym derivative is given by (compare Section 3 of Ref. [1])

$$\frac{d\mu_X}{d\mu_Y}(\xi) = \exp \{- \Sigma_{X/Y}(\xi)\}\quad (4)$$

where $\Sigma_{X/Y}: C(x_i) \to \mathbb{R}$

$$\Sigma_{X/Y}(\xi) = \frac{\xi(t)}{\xi(o)} \int_{\xi(o)}^{\xi(t)} dx G^{-2}(x) \left[ k(x) - f(x) \right] - \frac{t}{2} \int_0^t ds F'(\xi(s))$$

(5)

and $F: K \to \mathbb{R}$

$$F(x) = \frac{d}{dx} \left[ k(x) - f(x) \right] +$$

$$+ \left[ k(x) - f(x) \right] G^{-2}(x) \left[ k(x) + f(x) - \frac{d}{dx} G^2(x) \right].$$

(6)

As shown in [1], $-\Sigma_{X/Y}$ is just the density of the generalized entropy $S(\mu_Y \mid \mu_X)$ with respect to the measure $\mu_Y$, i.e.,

$$S(\mu_Y \mid \mu_X) = \int_{C(x_i)} d\mu_Y(\xi) \{- \Sigma_{X/Y}(\xi)\}.\quad (7)$$

2. A LOWER BOUND THEOREM

We define a local generalized entropy $\sigma: K \times K \times \mathbb{R}^+ \to \mathbb{R}$ by

$$\sigma(x_f, x_i, t) = - \int_{C(x_i)} d\mu_X(\xi) \delta(\xi(t) - x_f) \frac{d\mu_Y}{d\mu_X}(\xi) \ln \frac{d\mu_Y}{d\mu_X}(\xi)$$

such that

$$S(\mu_Y \mid \mu_X) = \int_K dx_f \sigma(x_f, x_i, t).\quad (8)$$

(9)

We remark that $\sigma$ is no information gain as $\mu_X$ and $\mu_Y$ are no probability measures on $(C(x_i, x_f), C(x_i, x_f) \cap \mathcal{B}(C([0, t])))$ where

$$C(x_i, x_f) = \{ \xi \in C(x_i) \mid \xi(t) = x_f \}.\quad (10)$$