A LOWER BOUND FOR THE TRANSITION PROBABILITY DENSITY OF A DIFFUSION PROCESS

ALEXANDER BACH

Institut für Theoretische Physik I der Universität Münster, Corrensstrasse, D-4400 Münster, W. Germany.

ABSTRACT. We give a lower bound for the transition probability density of a diffusion process in terms of the transition probability density of an auxiliary process.

1. INTRODUCTION

Let \((\Omega, F, \mu)\) be a probability space and \(\nu\) another probability measure on \((\Omega, F)\), which is equivalent to \(\mu\). The generalized entropy (information gain) of \(\mu\) with respect to \(\nu\) is defined by

\[
S(\mu | \nu) = - \int_{\Omega} \ln \left( \frac{d\mu}{d\nu} (\omega) \right) d\nu(\omega).
\]

In [1] we have applied this concept to measures induced by diffusion processes as follows. Let \(X_s, Y_s, s \in [0, t]\) one-dimensional diffusion processes with state space \(K \subset \mathbb{R}\), which are defined by the stochastic differential equations of an Itô type [2]

\[
dX_s = f(X_s) \, ds + G(X_s) \, dW_s, \quad X_0 = x_i \in K
\]

\[
dY_s = k(Y_s) \, ds + G(Y_s) \, dW_s, \quad Y_0 = x_j \in K.
\]

Here \(W_s, s \in [0, t]\), denotes the standard Wiener process and we assume that \(t < \min(t_X, t_Y)\), where \(t_X, t_Y\) denotes the explosion time of the process \(X_s\) and \(Y_s\), respectively.

We denote, by \(\mu_X, \mu_Y\), the measures induced on the measurable space \((C(x_i), C(x_i) \cap \mathcal{B}(C([0, t])))\) by \(X_s\) and \(Y_s\) respectively. Here \(C(x_i)\) is a subset of the space of continuous functions \(C([0, t])\) on \([0, t]\)

\[
C(x_i) = \{\xi \in C([0, t]) | \xi(0) = x_i\}
\]

and \(\mathcal{B}(C([0, t]))\) denotes the Borel field generated by the open subsets of \(C([0, t])\), with respect to the uniform norm.

Due to the fact that the diffusion terms, as well as the initial conditions of the processes \(X_s\) and

\[
\]
are identical, \( \mu_X \) is equivalent to \( \mu_Y \) and the Radon–Nikodym derivative is given by (compare Section 3 of Ref. \[1\])

\[
\frac{d\mu_X}{d\mu_Y}(\xi) = \exp \left\{ - \Sigma_{X/Y}(\xi) \right\}
\]

(4)

where \( \Sigma_{X/Y} : C(x_i) \to IR \)

\[
\Sigma_{X/Y}(\xi) = \int_{\xi(o)}^{\xi(t)} dx G^{-2}(x) \left\{ k(x) - f(x) \right\} - \frac{1}{2} \int_0^t ds F'(s)\left(\xi(s)\right)
\]

(5)

and \( F : K \to IR \)

\[
F(x) = \frac{d}{dx} \left\{ k(x) - f(x) \right\} + \\
+ \left\{ k(x) - f(x) \right\} G^{-2}(x) \left\{ k(x) + f(x) - \frac{d}{dx} G^2(x) \right\}
\]

(6)

As shown in \[1\], \(-\Sigma_{X/Y}\) is just the density of the generalized entropy \( S(\mu_Y | \mu_X) \) with respect to the measure \( \mu_Y \), i.e.,

\[
S(\mu_Y | \mu_X) = \int_{C(x_i)} d\mu_Y(\xi) \{-\Sigma_{X/Y}(\xi)\}.
\]

(7)

2. A LOWER BOUND THEOREM

We define a local generalized entropy \( \sigma : K \times K \times IR^+ \to IR \) by

\[
\sigma(x_f, x_i, t) = - \int_{C(x_i)} d\mu_X(\xi) \delta(\xi(t) - x_f) \frac{d\mu_Y}{d\mu_X}(\xi) \ln \frac{d\mu_Y}{d\mu_X}(\xi)
\]

(8)

such that

\[
S(\mu_Y | \mu_X) = \int_K dx_f \sigma(x_f, x_i, t).
\]

(9)

We remark that \( \sigma \) is no information gain as \( \mu_X \) and \( \mu_Y \) are no probability measures on \( (C(x_i, x_f), C(x_i, x_f) \cap \mathcal{B}(C([0, t])) \) where

\[
C(x_i, x_f) = \{ \xi \in C(x_i) | \xi(t) = x_f \}.
\]

(10)