GENERALIZED SYMMETRIES AND CONSTANTS OF MOTION OF EVOLUTION EQUATIONS

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ABSTRACT. It is shown that there exists a well defined one-to-one correspondence between constants of motion and generalized symmetries of an evolution equation iff the evolution equation admits a Lagrangian formulation. This correspondence is given explicitly.

1. INTRODUCTION

Noether's first theorem [1] gives a connection between the invariance of a variational integral and constants of motion of the corresponding Euler equation. However, this theorem and its generalizations (for references, see [2], §4) have certain disadvantages:

(i) Conservation laws are derived from symmetries of the variational integral rather than from symmetries of the corresponding Euler equation. Given an admissible operator of the latter it must be checked whether it is also admitted by the variational integral. (This need not always be the case; for example, the dilations define an admissible operator for the Laplace equation, but not for the corresponding variational integral.) Only then can we construct a conservation law (using a standard algorithm).

(ii) The inverse of Noether's theorem [3] provides a way of finding an admissible operator of Euler's equation given a conservation law, which however may be of only formal value as it might lead to trivial admissible operators (this actually occurs in [4]).

(iii) It is assumed explicitly that the equation under consideration is derivable from a variational principle. The question whether this is a necessary condition for a correspondence between symmetries and conservation laws is not posed (this remark also applies to [5], although other results in that paper, similar to the ones presented here, are derived by further analyzing Noether's theorem).

In the case of evolution equations we have overcome the above disadvantages and determined a well defined connection between symmetries and constants of motion, using a direct approach which is not based on Noether's theorem. Further, we do not assume a priori the existence of a Lagrangian formulation; it is interesting that this follows from our analysis (see Theorems 1 and 2). Essential in our approach is a theorem in [6], which characterizes the range of the Euler operator. A slightly stronger form of this theorem is given here.
Admissible LB Operators

In what follows we shall consider a general evolution equation of the form

\[ \Omega = u_t + K(x, u, u_1, u_2, ..., u_N) = 0, \]  

where \( u_j = \left( \frac{\partial}{\partial x} \right)^j u, \quad j = 0, 1, ..., n. \)  

The existence of a generalized symmetry of a given equation manifests itself by the existence of an admissible Lie–Bäcklund (LB) operator \[7\]. The most general LB operator associated with (1) is given by

\[ X(B) = B \frac{\partial}{\partial u} + (D_x B) \frac{\partial}{\partial u_x} + \sum_{i=1}^{\infty} (D^i B) \frac{\partial}{\partial u_i}, \]  

where \( B = B(x, t, u, u_1, ..., u_N), \quad N \) arbitrary, \( D \) is the total derivative with respect to \( x \)

\[ D \equiv D_x \equiv \frac{\partial}{\partial x} + u_1 \frac{\partial}{\partial u} + u_{t1} \frac{\partial}{\partial u_t} + u_2 \frac{\partial}{\partial u_1} + ... \]

and \( D_x \) is defined analogously. (Without loss of generality, we assume that \( B \) does not depend on \( t \)-derivatives since they can always be eliminated using eqn. (1).)

The LB operator \( X(B) \) is an admissible operator for eqn. (1) iff

\[ X(B) \Omega = 0, \quad \text{where} \quad \Omega = 0 \quad \text{when eqn. (1) and its differential consequences are assumed.} \]

The above is denoted as

\[ X(B) \Omega \big|_{\Omega = 0} = 0. \]  

Equation (4) provides an algorithm for finding \( B \) (which may be trivial) as the solution of a system of linear overdetermined equations.

An important special class of LB operators is the class of Lie (point) operators \[2\]. The most general such operator, associated with (1), is given by

\[ Y = \xi \frac{\partial}{\partial x} + \tau \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial u}, \]  

where \( \xi, \tau, \eta \) are functions of \( x, t \) and \( u \) only. The operator \( Y \) can be written in the form (3) by the equivalence \[2, \S 1.5.3\].

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